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THE SOCIAL PRODUCTION OF MATHEMATICS FOR TEACHING

ABSTRACT

This chapter aims to show the impact of culture on the learning of mathematics and consequently that studies of mathematics for teaching require strong theoretical frameworks that foreground the relationship between culture and pedagogy. For this purpose, we describe two different research projects in Southern Africa, each focused on the notion of mathematics for teaching. The first study analyses teacher learning of the mathematical concept of limits of functions through participation in a research community in Mozambique, and is framed by Chevallard's anthropological theory of didactics. The second, the QUANTUM project, studies what and how mathematics is produced in and across selected mathematics and mathematics education courses in in-service mathematics teacher education programmes in South Africa, and is shaped by Bernstein's theory of pedagogic discourse. We argue that separately and together these two studies demonstrate that mathematics for teaching can only be grasped through a language that positions it as structured by, and structuring of, the pedagogic discourse (in Bernstein's terms) or the institution (in Chevallard's terms) in which it 'lives'.

INTRODUCTION

Shulman (1986, 1987) posited the notions of *subject matter knowledge* (SMK), *pedagogical content knowledge* (PCK) and *curriculum knowledge* (CK), as critical categories in the professional knowledge base of teaching. In so doing, he foregrounded the centrality of disciplinary or subject knowledge, and its integration with knowledge of teaching and learning, for successful teaching. The past two decades have witnessed a range of studies related to SMK and an emphasis of research on PCK, many focused on mathematics (e.g., Ball, Bass and Hill, 2004; Ball, Thames and Phelps, 2007). As a consequence, a new discourse is emerging attempting to mark out *mathematics for teaching* as a distinctive or specialized form of mathematical knowledge produced and used in the practice of teaching. As noted in Adler & Davis, (2006), this discourse is fledgling.

In this chapter we describe two different research projects in Southern Africa each focused on the notion of *mathematics for teaching*. We foreground the social epistemologies that informed and shaped these studies: Chevallard's anthropological theory of didactics (Chevallard, 1992, 1999) and Bernstein's theory of pedagogic discourse (Bernstein, 1996, 2000), and illuminate their critical

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role in each study. We argue that separately and together these two research projects demonstrate that *mathematics for teaching* can only be grasped through a language that positions it as structured by, and structuring of, the pedagogic discourse (in Bernstein's terms) or the institution (in Chevallard's terms) in which it 'lives'. From this perspective, mathematics is learned for some purpose, and within teacher education, this would be for mathematics teaching, and/or becoming a mathematics teacher. There are thus limits to the appropriateness of the use of general categories like PCK and SMK, as well as the distinctions between them.

MOZAMBICAN TEACHERS' RESEARCHING THE LIMIT CONCEPT

We begin with a study of teacher learning through participation in a research community in Mozambique, motivated by the desire to impact on teachers' knowledge of advanced mathematical concepts. The study drew inspiration for its questions from Chevallard's notions of personal and institutional relations to concepts, and from his elaborated anthropological theory of didactics for framing and interrogating the notion of *mathematics for teaching* in this study. It shows how the institutional relation to the mathematical concept of limits of functions, as well as each teacher's position within the new institution influenced the development of a new personal relation to this concept. However, the weight of strong institutions such as Mozambican secondary school and Pedagogical University hindered the development of a more elaborated relation to the mathematical concept of limit that allows the challenging of these two strong institutions' relation to this concept. This opens up questions about both SMK and PCK and their inter-relation, particularly in teacher education practice.

Starting Point of the Study

In Mozambican didactic institutions, the teaching of limits of functions typically has two components: a formal component, the ε-δ definition, derived from within mathematics that students are sometimes asked to memorise; and a procedural component, the calculation of limits using algebraic transformations. Mozambican teachers study the limit concept in these didactic institutions, secondary schools and university. As a consequence, their mathematical knowledge of limits is reduced to these two aspects (Huillet & Mutemba, 2000). Their teaching mirrors the way they have been taught as students, and thus, the secondary schools' routine for teaching limits. This study started from reflection on the conditions for changing these institutional routines. Considering that teachers are the main actors in the didactical relation in the classroom, teaching limits in a more elaborated way would only be possible if teachers develop their mathematical knowledge of this concept. This led to the following questions:

- How could limits of functions be taught in Mozambican secondary schools so that students not only learn to calculate limits but also give meaning to this concept?

- What kind of knowledge does a teacher need to teach limits in schools in that way?
- How could Mozambican secondary school teachers acquire this knowledge?

These questions have been addressed in this study through the lens of the Chevallard's anthropological theory of didactics.

Chevallard's Anthropological Theory of Didactics

The anthropological theory of didactics (ATD) locates mathematical activity, as well as the activity of studying mathematics, within the set of human activities and social institutions (Chevallard, 1992). It considers that "everything is an object" and that an object exists if at least one person or institution relates to this object. To each institution is associated a set of "institutional objects" for which an institutional relation, with stable elements, is established.

An individual establishes a personal relation to some object of knowledge if s/he has been in contact with one or several institutions where this object of knowledge is found. S/he is a "good" subject of an institution relative to some object of knowledge if his/her personal relation to this object is judged to be consistent with the institutional relation. For example, in this study, the relation that Mozambican mathematics teachers established with the limit concept was shaped by the relationship to this concept in the institutions in which they learned it. For most teachers, this contact occurred in Mozambican institutions (secondary school as students, university as students, and as secondary school teachers). The institutional relation to an object of knowledge can be analysed through the social practices involving this object inside the institution. Chevallard (1999) elaborates a model to describe and analyse these institutional practices, using the notion of praxeological organisation or, in the case of mathematics, mathematical organisation. The first assumption of this model is that any human activity can be subsumed as a system of tasks (Chevallard, 1999; Bosch and Chevallard, 1999). Mathematics, as a human activity, can therefore be analyzed as the study of given kinds of problematic tasks.

The second assumption of this theory is that, inside a given institution, there is generally one technique or a few techniques recognized by the institution to solve each kind of task. Each kind of task and the associated technique form the *practical block* (or know-how) of a *mathematical organisation* (MO). For example, in Mozambican secondary schools, students are taught to calculate limits using algebraic transformations. A specific algebraic transformation is associated to each kind of limit, constituting the practical block of a specific MO. Other kinds of tasks could be: to read limits from a graph, to sketch the graph of a function using its limits, to demonstrate the limit of a function using the definition, etc. These kinds of tasks are hardly used in Mozambican secondary schools, but can be found in other institutions, for example in secondary schools or universities in other countries. Students are then expected to solve each of these tasks using a specific technique. The institutional relation to an object is shaped by the set of tasks to be performed, using specific techniques, by the subjects holding a specific position inside the institution. In an institution, a specific kind of task T is usually solved using only one technique τ. Most of the tasks become part of a routine, the task/technique practical blocks $[T, \tau]$ appearing to be *natural* inside this institution.

The third assumption of the theory of mathematical organisations is that there is an ecological constraint to the existence of a technique inside an institution: it must appear to be understandable and justified (Bosch & Chevallard, 1999). This is done by the technology θ, which is a rational discourse to describe and justify the technique. This constraint can be interpreted at two levels. At the students' level, it means that students should be able to understand the technique. At the mathematics level, we must ensure that the technique is "mathematically correct" with reference to scholarly knowledge[1]. These ecological constraints can sometimes lead to a contradiction, given that the ability of students to understand will be constrained by their development and previous knowledge. It can be difficult for a technique to be both understandable and justified at the same time.

The technology θ itself is justified by a theory Θ, which is a higher level of justification, explanation and production of techniques. Technology and theory constitute the *knowledge block* $[\theta,\Theta]$ of a MO. According to Chevallard (1999), the technology-theory block is usually identified with *knowledge* [un savoir], while the task-technique block is considered as *know-how* [un savoir-faire].

The two components of an MO are summarized in the diagram below.

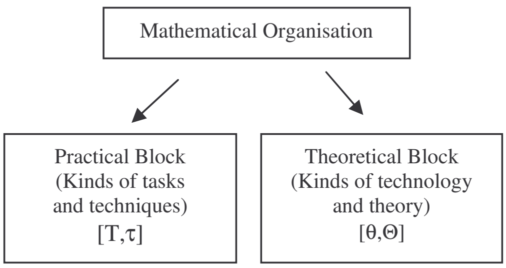

Figure 1. Mathematical organisation

An MO around a particular kind of task in a certain institution is specific. For example, calculating the limit of a rational function when x goes to infinity by factorisation and cancellation is a specific MO. The corresponding technology would be, for example, the theorems about limits, and the corresponding theory the demonstration of these theorems using the ε-δ definition. The integration of several specific MOs around a specific technology gives rise to a local MO. For example, calculating several kinds of limits using algebraic transformation constitutes a local

[1] We mark out mathematical knowledge intentionally in order to signal that what counts as scholarly mathematical knowledge is not unproblematic.

MO. In the same way, the integration of several local MOs around the same theory gives rise to a regional or global mathematical organization.

In order to teach a mathematical organization, a teacher must build a *didactical organisation*[2] (Chevallard, 2002). To analyse how a didactical organisation enables the set up of a mathematical organisation, we can first look at the way the different moments of the study of this MO are settled in the classroom. Chevallard (2002) presents a model of six moments of study. They are the following: *first encounter* with the MO, *exploration of the task* and *emergence of the technique*, construction of the *technological-theoretical bloc*, *institutionalisation*, work with the MO (particularly *the technique*), and *evaluation*[3]. The order of these moments is not a fixed one. Depending on the kind of didactical organisation, some of these moments can appear in a different order, but all will probably occur. For example the study of mathematical organisations at university level is often divided in theoretical classes and tutorials. The theoretical block is presented to students in lectures, as already produced and organised knowledge, and tasks are solved using some techniques (practical block) during tutorials. In that way there is a disconnection between the theoretical component of the organisation and its applications. This is what happens in Mozambique with limits of functions. At the university level, the ε-δ definition and the theorems about limits and their demonstrations using this definition are usually taught in theoretical classes, while tutorials are dedicated at calculating limits using algebraic transformations (Huillet, 2007a). In that case, the reasons why the theory exists gets lost. And so we see the institution as structuring of, and being structured by, the particular mathematics in focus.

The Use of Anthropological Theory of Didactics in this Study

The anthropological theory of didactics (ATD) has been used in this study to analyse the teachers' personal relation to this concept and its evolution through their work within a new institution, using different aspects of *mathematics for teaching* limits.

In the first place, ATD has been used as a tool for analysing the institutional relation of Mozambican didactic institutions to the limit concept, in particular the secondary school institution and the Pedagogical University where most of mathematics teachers are trained. For each of these institutions, the practical block and the knowledge block of the mathematical organisation related to limits of functions have been analysed through the examination of the syllabus, the national examinations (secondary school), worksheets used in secondary schools (there is no textbook for this level in Mozambique), textbooks used at the Pedagogical University and the exercise book of a student. This analysis highlighted a

[2] We note here that didactical organisations are specific to certain topics or contents in mathematics.

[3] There is an interesting similarity between these moments of study and the interpretation of Hegelian moments of judgement in pedagogic discourse as described by Davis (2001), and referred in Adler and Pillay (2007).

dichotomy between two regional mathematical organisations: the algebra of limits based on the ε-δ definition, and the existence of limits, based on algebraic transformations to evaluate limits. This dichotomy, which also exists in other secondary schools in other countries (Barbé, Bosch, Espinoza & Gascón, 2005) and is explained by the nature of the limit concept, seems to be exacerbated in the Mozambican case. This can explain the limited personal relation to limits of Mozambican teachers (Huillet, 2005a).

Secondly, anthropological theory of didactics was used to design the research methodology. Considering the institutional relation previously described and how it strongly shaped teachers' personal relations to this concept, this personal relation could only evolve if teachers were in contact with this concept through a new institution where this concept lived in a more elaborated way[4]. Other institutional or personal constraints could influence the usual way of teaching limits in schools. The argument was that their personal relation did not allow them to challenge the institutional routines. The evolution of their knowledge was a necessary, although not sufficient, condition for any change of the way of teaching limits in Mozambican secondary schools. Consequently a new institution was set up, where four final-year student-teachers from Pedagogical University researched some aspects of the limit concept and shared their findings in periodic seminars. The researcher was both supervisor of the teachers' individual research and facilitator of the seminars.

In the third place, ADT was used to analyse a mathematics teacher's task(s) when planning a didactical organisation, using Chevallard's model of the moments of study (Chevallard, 2002). This allowed the development of a general framework for describing the knowledge needed by a teacher to perform these tasks. It includes scholarly mathematical knowledge of the MO, knowledge about the social justification for teaching this MO, how to organise students' first encounter with this MO, knowledge about the practical block (tasks and techniques) using different representations, knowledge on how to construct the theoretical block according to learners' age and previous knowledge, and knowledge about students' conceptions and difficulties when studying this MO. This description of *mathematics for teaching*[5] was used to define research topics for the teachers involved in the research group. In line with the overall approach in this study, within these aspects of *mathematics for teaching* limits, the boundary between SMK and PCK as developed by Shulman is blurred. Rather, and this is elaborated further below, each aspect has two components, a mathematical and a pedagogical component. Some aspects are more mathematical, some others more pedagogical, but they are necessarily merged in the human activity of mathematics teaching.

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[4] Obviously, it cannot be claimed that a change in teachers' personal relation would automatically result in a change of their way of teaching limits at school.
[5] The expression *mathematics for teaching* to design the knowledge needed by a mathematics teacher is the same as defined by Ball et al. (2004) and Adler and Davis (2006).

The Evolution of Teachers' Knowledge through the New Institution

The new institution set up for this study was a research group, where four teachers, honours students at the Pedagogical University, researched different specific aspects of the limit concept and shared their findings in periodical seminars. The researcher was their supervisor and the facilitator of the seminars. The evolution of these teachers' knowledge through the new institution was analysed in detail for five aspects, or sub-aspects, of *mathematics for teaching* limits in schools: how to organise students' first encounter with limits of functions, the social justification for teaching limits in secondary schools, the essential features of the limit concept (part of the scholarly mathematical knowledge), the graphical register (part of the practical block) and the ε-δ definition (also part of the scholarly mathematical knowledge). For each of these aspects, categories were defined both for teachers' mathematical knowledge (ranked in several degrees from "knowing less" to "knowing more") and for teachers' ideas about teaching, related to this aspect (ranked again in several degrees from "being close to the secondary school institutional relation to limits" to "challenging this institutional relation"). An example of each of these can be found in Appendix 1, where the co-presence of both mathematical and teaching knowledge is evident. These further illuminate that learning of mathematics through the range of tasks in this research institution was for the purposes of teaching mathematics.

In this chapter we will not detail the methodology used to collect and analyse data in the study. We only present some results that help understand the role of the new institution in the development of a new personal relation to limits and discuss the weight of this institution in comparison with strong Mozambican didactic institutions as are the Secondary School and Pedagogical University. We focus on two teachers, selected because they represent two extreme situations in relation to their teaching experience and to their position within the group: Abel[6], an experienced teacher who had taught limits in school for years; and David, the youngest teacher in the group, with very little teaching experience. The evolution of these two teachers' personal relation to limits during the research process, for the five aspects selected for the study, is presented in Table 1 below.

Table 1. Evolution of Abel' and David's Personal Relation to the Limit Concept According to Five Aspects of Mathematics for Teaching Limits

	Categories	Abel	David
First Encounter	FE-MK1 to FE-MK2	FE-MK1→FE-MK2	FE-MK1 → FE-MK2
	FE-T1 to FE-T6	FE-T2 → FE-T4	FE-T1 → FE-T5
Social Justification	SJ-MK1 to SJ-MK4	SJ-MK2 → SJ-MK4	SJ-MK1 → SJ-MK4
	SJ-T1 to SJ-T3	SJ-T1 → SJ-T1	SJ-T1 → SJ-T3

[6] These are pseudonyms

Essential Features	EF-MK1 to EF-MK4	EF-MK1→ EF-MK4	EF-MK2→ EF-MK4
	EF-T1 to EF-T2	EF-T1→EF-T2	EF-T1→EF-T2
Graphical Register	GRRR to GRR6	GRR1 → GRR3	GRR2 → GRR5
	GRRS to GRS6	GRS2 → GRS2	GRS3 → GRS6
	GR-T1 to GR-T3	GR-T1 → GR-T2	GR-T1 → GR-T3
ε-δ Definition	D-MK1 to D-MK4	D-MK1 → D-MK3	D-MK1 → D-MK2
	D-T1 to D-T4	D-T1 → D-T3	D-T4 → D-T2

FE-MK *Mathematical knowledge about the first encounter with the limit concept.*
FE-T *Ideas about teaching related to the first encounter with the limit concept.*
SJ-MK *Mathematical knowledge about the social justification.*
SJ-T *Ideas about teaching related to the social justification.*
EF-MK *Mathematical knowledge about essential features.*
EF-T *Ideas about teaching related to essential features.*
GRRR *Knowledge about how to read limits from graphs.*
GRRS *Knowledge about how to represent a limit on a graph.*
GR-T *Ideas about teaching related to the graphical register.*
D-MK *Mathematical knowledge about the definition.*
D-T *Ideas about teaching related to the definition.*

This table shows that while both teachers' personal relation to limits evolved, this was uneven, particularly for the two last aspects of the limit concept: the use of the graphical register and the ε-δ definition. In this chapter we focus on these two aspects.

In the wider study, Huillet (2007b) shows the limited evolution of all four teachers' knowledge about the graphical register, and explained this in relation to the general difficulty that the teachers had in working with graphs. The teachers in this study did not display deep understanding of basic mathematical knowledge such as the concept of function, and the use and interpretation of graphs in general. Nevertheless, the evolution of Abel's and David's knowledge about the use of the graphical register for studying limits was very different from each other, with David's knowledge about this aspect evolving more than Abel's (as well as the other teachers' in the study). We suggest two explanations for this uneven outcome. Firstly, this aspect was directly linked to David's research topic (Applications of the limit concept in mathematics and in other sciences). Secondly, and this explanation is more speculative, David used the interviews as a means for learning. By positioning himself more as a student than as a teacher, David was able to take advantage of each opportunity for learning, by asking questions and attempting to solve more tasks. In contrast, Abel did not try to solve many graphical tasks. He assumed more of a teacher's position and thus one who should already know. As a consequence, he did not engage in the interviews in ways that could have enabled his knowledge about the use of graphs for teaching limits to evolve.

The ε-δ definition belongs to the scholarly mathematical knowledge and, like the graphical register, requires a deep understanding of basic mathematical concepts. Furthermore, it is intrinsically difficult (Huillet, 2005b) and it is part of the syllabus of Mozambican secondary schools. At the beginning of the research process, none of the teachers could explain this definition, and their understanding of it evolved slightly during the study. Again, Abel's and David's ideas about teaching this definition in secondary schools evolved differently, curiously in opposite directions. At the beginning of the research process, Abel, the experienced teacher who had taught this definition in schools, argued that it was right that it be taught in school. By the end of the study, he had reached the conclusion that it was not appropriate to teach this definition at secondary school level: students were not able to understand it in that form. In contrast, David was initially inclined not to teach the definition. At the end, however, he was willing to teach it while acknowledging students' difficulties in understanding this definition.

Let's analyse the evolution of how these two teachers' positioned[7] themselves within the new institution, that is, within the research group and so with the possibilities for their relations to the limit to evolve. At the beginning of our work together, Abel positioned himself as an experienced teacher. During the seminars, he volunteered to explain some aspects of limits to his colleagues, particularly the ε-δ definition, trying to show that he had mastered this topic. During the first interview, he constantly referred to what was done in schools when teaching this concept, showing that he knew the syllabus and the way limits are usually taught in secondary schools. However, he faced difficulties during his explanations to his colleagues and felt 'ashamed' about it, as he told the researcher during the second interview. He then faced difficulties during his research study - an experiment in a secondary school. He experienced these difficulties as his failure as a teacher, and not as the result of the research, or as a researcher. He also became aware that he had been teaching limits in school, in particular the ε-δ definition and L´Hôpital's Rule, in ways that were problematic for his students. This reflection on his practice, although very hard for him, offers an explanation as to why, at the end of the research process, Abel said that the ε-δ definition should not be part of the secondary school syllabus.

In contrast, David's initial position within the research group was of learner-teacher, a university student teaching as he completed his studies. During the first interview, he analysed the way limits are usually taught in Mozambican secondary schools as a student who did not understand the ε-δ definition. He did not participate much in the discussion during the first seminars, giving way to his more experienced colleagues. However, he was able to argue with them in the last seminars. The end of the research process coincided with the conclusion of his teacher training course, and it seems that at that point, he then positioned himself more as a teacher than as a student. He was thus anticipating the institution where

[7] What we mean by 'positioned' here is the way in which this particular teacher related to both the researcher and others in the group.

he was going to teach limits i.e., the secondary school institution. He knew that secondary school students were not able to understand this definition, but at the same time that it is part of the Mozambican Grade 12 syllabus. He remembered studying it in that grade. It is arguable, that as a prospective teacher, the weight of the institution he was moving to became more influential in his thinking.

The analysis of the evolution of these two teachers' personal relation to limits of functions related to two critical aspects of this concept shows that these teachers experienced the weight of the two institutions in different ways, particularly where relations to the limit concept was in conflict.

Institutional Strengths and Weaknesses

Chevallard's anthropological theory of didactics points out the importance of institutional relations to an object of knowledge and how an individual's personal relation to this object is shaped by the institutions' relations where this individual met this object. The study of the personal relation of four Mozambican student-teachers, who had mainly[8] been in contact with the limit concept through Mozambican institutions, showed how their personal relation to limits at the start of the study was consistent with the Mozambican Secondary School's and Pedagogical University's institutional relations to this concept. It also showed that this personal relation evolved during their contact with this object of knowledge through another institution, the research group, holding a different institutional relation. However, this study also pointed out some limitations in the evolution of this knowledge.

These results lead to the following questions:

- Why, at the end of the work within the research group, was the teachers' personal relations to limits of function not fully consistent with the relation of the new institution?
- How could the new institution be modified so as to enable teachers to learn more about limits, according to the expected (and more elaborated) personal relation?

Elements of answers to these questions have been given in the previous section. One of them is the lack of basic understanding of some mathematical concepts that hindered the evolution of teachers' knowledge, especially the mathematical components of this knowledge. It seems that, in these cases, more direct engagement with these aspects of the limit, supported by explanations and systematic solution of tasks was necessary for teachers to overcome their difficulties. This is what happened with David. During the third interview he tried to solve many graphical tasks, asking questions and drawing on the researcher's explanations to solve them. He thus engaged with the limit concept in various and new ways.

[8] One of the teachers also studied in a university in East Germany

The mathematical component in the new institution was apparently not strong enough. In the first case, the researcher did not anticipate the extent of the weakness of the teachers' knowledge of basic mathematical concepts. For example, she knew that the teachers were not used to using graphs in the study of limits, but she did not imagine that they would have so many difficulties working with graphs in general. For example, they sometimes confused *x*-values and *y*-values (or the two axes), or a limit with the maximum of the function (Huillet, 2007b). Secondly, the researcher was reluctant to play the role of a teacher within the group, because she wanted to observe how the teachers' personal relation to limits evolved through research. She felt that teaching them would influence the results of her research.

With regard to the more pedagogical component of the teachers' personal relations to limit, we already saw that, during the research process, all teachers changed their ideas about teaching the ε-δ definition in secondary schools. While the experienced teacher said that he would not like to teach the definition any more, the other three teachers argued that this definition should be taught in schools. This evolution was explained by the weight of the secondary school institutional relation to limits: they were now positioned as teachers and not as students, as at the beginning of the research process. This suggests another weakness of the research group as an institution. Relative to well established institutions such as the Mozambican Secondary School and the Pedagogical University, institutions with a strong tradition of teaching and well established routines, the research group appears as a very weak institution. That this institution enabled the teachers to become aware of strong gaps in the teaching of limits in secondary schools and at university does not necessarily imply that they will be able to stand up against strongly institutionalised routines. Organising students' first encounter with limits in a different way, introducing different kinds of tasks, for example graphical tasks or tasks to link limits with other mathematical or other sciences concepts, do not mean going against the secondary school syllabus, but adding something to it. Not to teach the ε-δ definition, even knowing that the students will not understand it, is a bigger step to take because this definition is part of the syllabus. It can be seen as an act of rebellion against the institution. Elsewhere, Huillet (2007a) has argued that these research outcomes emerging as they are from the Mozambican context and through a study that placed mathematics at its centre open up important questions about the literature in mathematics education on teachers-as-researchers. In the "teachers as researchers" movement, teachers usually studied some pedagogical aspect of their teaching, taking the mathematical content for granted; this did not allow them to challenge the content of their teaching. This can also be seen as the result of the dichotomy between mathematics and pedagogy in teacher education. This dichotomy is reproduced in Shulman's distinction between SMK and PCK, as if SMK were some kind of 'universal mathematical knowledge' and PCK mathematical knowledge specific for teaching. However, his description of SMK's substantive and syntactic structures contradicts this separation.

> The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organised to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established (Shulman, 1986, p.9).

The syntactic structure of the discipline is important for teachers to engage in mathematics in a way that enables the construction of new didactical organisations. This study shows that the teachers involved in the research group did not grasp the syntactic structures of the limit concept during their training, but were only able to lead with substantive structure. From our perspective, this is not sufficient to teach limits in a way that challenges institutional routines, so making it comprehensible to students. We can then ask the question: does the syntactic structure of mathematics belong to SMK or PCK? We argue that this distinction is not appropriate and that, in teacher education, mathematics should live in a way that enables reflection at the same time on the mathematical and pedagogical aspects of the content to be taught. Where, when and how, then, in teacher education (particularly in professional development) practice, are teachers to have opportunities for further engagement with both syntactic and substantive aspects of the limit function. Could a research institution be strengthened so as to offer teachers further possibilities for elaborating their knowledge of limits of functions, and if so, how? The institution of mathematics teacher education itself – its objects and tasks, in Chevallard's terms - are in focus in the QUANTUM research project. In the next section we describe aspects of QUANTUM, foregrounding the theoretical resources drawn on to enable us to 'see' this "inner logic of pedagogic discourse and its practices" (Bernstein, 1996, p. 18), and specifically how it comes to shape *mathematics for teaching* in teacher education practice.

THE QUANTUM RESEARCH PROJECT IN SOUTH AFRICA

QUANTUM is the name given to a research and development project on quality mathematical education for teachers in South Africa. The development arm of QUANTUM focused on **qu**alifications for **t**eachers **u**nderqualified in **m**athematics (hence the name) and completed its tasks in 2003; QUANTUM continues as a collaborative research project. Between 2003 and 2006, the QUANTUM project has studied selected mathematics and mathematics education courses offered in higher education institutions as part of formalised (i.e., accredited) mathematics teacher education programmes for practicing teachers in South Africa. Our analysis of these courses led to deeper insights into and understanding of what and how *mathematics for teaching* comes to 'live' in such programmes. We drew on, and elaborated, a set of theoretical resources from Bernstein in our study. These are our focus in this section of the chapter.

Starting Point of the Study

An underlying assumption in QUANTUM is that mathematics teacher education is distinguished by its dual, yet thoroughly interwoven, objects: *teaching* (i.e., learning to *teach* mathematics) and *mathematics* (i.e., learning *mathematics for teaching*). It is these dual objects that lead to what is often described as the subject-method tension. Others describe this as one of the dilemmas in teacher education (Adler, 2002; Graven, 2005). The inter-relation of mathematics and teaching is writ large in *in-service* teacher education (INSET) programs (elsewhere referred to as professional development for practicing teachers) where new and/or different ways of knowing and doing school mathematics, new curricula, combine with new and/or different contexts for teaching. Such are the conditions of continuing professional development for practicing teachers in South Africa. The past ten years saw a mushrooming of formalised programs for practicing teachers across higher education institutions in South Africa, in particular, Advanced Certificates in Education (ACE) programs[9]. The ACE qualification explicitly addresses the inequities produced in apartheid teacher education, where black teachers only had access to a three-year diploma qualification. As a result, most ACE programs are geared to black teachers, at both the primary and secondary levels. Many of these are focused on the content of mathematics and constituted by a combination of mathematics and mathematics education courses. Debate continues as to whether and how these programs should integrate or separate out opportunities for teachers to (re)learn *mathematics* and to (re)learn how to *teach*.

A consequent assumption in QUANTUM is that however the combinations are accomplished, both mathematics and teaching as activities and/or discourses are always simultaneously present in all components of such programmes. Moreover, their interaction within pedagogic practice will have effects. This latter assumption is derived from a social epistemological approach to knowledge (re)production in pedagogic practice, and motivated by the work of Basil Bernstein, specifically how he deals with the conversion or translation of knowledges into pedagogic communication. And it is this orientation that leads us to reframe the broad problematic discussed as the following research question: What is constituted as *mathematics for teaching* in formalised practicing mathematics teacher education practice in South Africa, and how is it so constituted[10]?

Bernstein's Theory of the Pedagogic Device and Related Orientations to Knowledges

In the introduction to this chapter, we noted that Chevallard and Bernstein share a social orientation to knowledge. Both hold that rigour in educational (or didactics) research is a function of coherence between an overarching theoretical orientation, research questions and methodology. For Chevallard, a didactic organisation needs to be built to teach a mathematical organisation. The reciprocal effects of this are

[9] The ACE (formerly called a Further Diploma in Education – FDE) is a diploma that enables teachers to upgrade their three-year teaching diploma to a four-year diploma. The goal is to provide teachers with a qualification regarded as equivalent with a four-year undergraduate degree.

[10] In Chevellard's terms, the question would be: what is the institutional relation to mathematics that is set up and how does it function?

inevitable. Bernstein too sees knowledges in school, or any pedagogic context, as structured by pedagogic communication. His theory of the pedagogic device describes a set of principles and rules that regulate this structuring. It is these that we have brought to bear on our investigation into *mathematics for teaching* in teacher education practice.

For Bernstein, the principles of the transformation of knowledge in pedagogic practice are described in terms of the 'pedagogic device' (Bernstein, 2000). The pedagogic device is an assemblage of rules or procedures via which knowledges are converted into pedagogic communication[11]. It is this communication (within the pedagogic site) that acts on meaning potential. That is, pedagogic discourse itself shapes possibilities for making meaning, in this case of *mathematics for teaching*. The pedagogic device is the intrinsic grammar (in a metaphoric sense) of pedagogic discourse, and works through three sets of hierarchical rules.

Distributive rules regulate power relations between social groups, distributing different forms of knowledge and constituting *different orientations to meaning* (Bernstein refers to pedagogic identities). In simpler terms, the regulation of power relations in pedagogic practice effects who learns what. Whereas for Chevallard, orientations to meaning lie in 'institutional and personal relations' to a concept, the distributive rule brings social structuring *effects* to the fore, a function of Bernstein's concern with educational inequality and its social (re)production.

Recontextualisation rules regulate the formation of specific pedagogic discourse. In any pedagogic practice *knowledges are delocated, relocated and refocused*, so becoming something other. In the context of QUANTUM, the recontextualising rule at work regulates how mathematics and teaching, as a discipline and a field respectively, are co-constituted in particular teacher education practices. Here there is further resonance with didactic transposition, and with Chevallard's notion of institutionalisation, particularly the effects of strong and weak institutions on changing practices. The recontextualising rule is possibly the most well known and used element of Bernstein's work, and elaborated through the concepts of classification and framing. Classification refers to "the relations between categories" (2000, p. 6)[12], and how strong or weak are the boundaries between categories (e.g., discourses or subject areas in the secondary school) in a pedagogic practice. Framing refers to social relations in pedagogic practice, and who in the pedagogic relation controls what (2000). For our purposes in this chapter, the issue is whether and how mathematics and teaching as two domains are insulated from each other or integrated and then through what principles. The way knowledges are classified and framed, in any educational practice, the varying strength or weakness of the insulations, will constitute a range

[11] In Chevallard's terms, this transformation occurs in the setting up of the didactical organisation.
[12] For Bernstein, boundary maintenance is through power and changing or weakening the insulation between categories will reveal power relations – and so be contested (p.7).

of pedagogic modalities[13] and shape what comes to be transmitted[14]. In particular, they will impact on what comes to be *mathematics for teaching*.

Acquisition, in Bernstein's terms, is elaborated by what he refers to as 'recognition' and 'realisation'. In any pedagogic setting, learners need to recognise what it is they are to be learning, and further, they need to be able to demonstrate this by producing (realising) what is required – what he refers to as a 'legitimate text'[15]. Recognition and realisation link with the third set of rules operating within the pedagogic device. *Evaluative rules* constitute specific practices – *regulating what counts as valid knowledge*. For Bernstein, any pedagogic practice "transmits criteria" (indeed this is its major purpose). Evaluation condenses the meaning of the whole device (2000), so acting (hence the hierarchy of the rules) on recontextualisation (the shape of the discourse) that in turn acts on distribution (who gets what). What comes to be constituted as *mathematics for teaching* (i.e., as opportunities for learning *mathematics for teaching*) will be reflected through evaluation and how criteria come to work.

Despite the significance of evaluation in this theory, and in contrast to recontextualising rules, Bernstein's evaluative rules are not elaborated. Much of the pedagogical research on teacher education that has worked with Bernstein's framework focused on his rules for the transformation of knowledge into pedagogic communication, and particularly the distributive and recontextualising rules of the pedagogic device (e.g., Ensor, 2001, 2004; Morais, 2002). These studies foreground an analysis of classification and framing in a particular pedagogic modality, and related recognition and realisation rules that come to play. Ensor's study of mathematics prospective teacher education and its recontextualisation in the first year of teaching has advanced our understanding of the what, how and why of recontextualisation across sites of practice (university and school). The study argues that the 'gap' between what is taught in a programme for prospective teachers, and the practice adopted by teachers in their first year of teaching is not simply a function of teacher beliefs on the one hand, or constraints in schools on the other. The gap is explained through the principle of recontextualisation. The privileged pedagogy enacted in the teacher education programme was unevenly accessed by the teachers in her study. Ensor, drawing on Bernstein, shows how this distribution was a function of what and how criteria for the privileged practice were marked out, and so what teachers were or were not

[13] Bernstein describes two contrasting educational codes – ideal types – formed by strong and weak classification. A collection code has strong classification and strong framing; in contrast, an integrated code has weak classification and framing. In the latter boundaries between contents and between social relations are both weak.

[14] As Graven (2002) explains, "in educational terms, Bernstein's use of the terms 'transmitter' and 'acquirer' may seem pejorative. However, he uses them throughout various pedagogic models and they are merely sociological labels for descriptive purposes. They should therefore not be interpreted to imply transmission pedagogies". (Ch. 2, p.28).

[15] In Chevallard's terms, when learners are able to produce the legitimate text, they show that their personal relation fits the institutional relation (that they are "good subjects" of the institution).

able to recognise as valued mathematical practice, and then realise this in their school classrooms. Morais' work focused on primary science, and tackles the phenomenon of primary teachers not being subject specialists. She argues that because of their weaker science knowledge base, the pedagogic modality in their teacher education should combine strong classification with weakened framing. In this way primary science teachers can be offered an enabling set of social relations within which to engage further learning of science. Science content needs to be clearly bounded and visible (i.e., strongly classified), and structured by sequencing and pacing to suit primary teacher interests and needs (weakly framed).

However, as Davis has argued (Davis, 2005) a focus on classification and framing, while productive, backgrounds the special features of the content to be acquired. Even in Morais' study, the *specificity* of the science to be learned by primary teachers remains in the background. The concern with mathematical production in teacher education has thus led the QUANTUM project to focus instead on evaluation, and on the criteria for the production of legitimate (mathematical) texts. This required elaboration of the evaluative rule. Before we describe the methodology we have developed and used in QUANTUM to foreground the content to be acquired, one additional aspect of Bernstein's work requires discussion.

Mathematics and Teaching as Differing Domains of Knowledge.

Bernstein (2000) provides conceptual tools to distinguish different forms of knowledge and so to interrogate mathematics and teaching. In the first instance, he distinguished vertical and horizontal discourses, the criteria for which are forms of knowledge, and the most significant of which is whether knowledge is organised hierarchically or segmentally (2000). But, as he argues, this broad distinction does little to assist with understanding discourses in education, and ensuing issues of pedagogy. Education (and so too mathematics education) invokes a wide range of vertical discourses. He thus developed further distinctions, insisting that while these were accompanied by additional conceptual apparatus, they were important analytically.

For Bernstein, within vertical discourses we can distinguish between hierarchical and horizontal knowledge structures; and within the latter, between strong and weak grammars. Different domains of knowledge are differently structured and have different grammars. The natural sciences have hierarchical knowledge structures and strong grammars. They have "explicit conceptual syntax" and so recognition of what is and is not physics, for example, is apparent. Development is seen as "the development of theory which is more general and more integrating than previous theory" (2000, p. 162). The social sciences (hence education), in contrast, have horizontal knowledge structures, where development proceeds through the introduction of "new languages" that "accumulate" rather than integrate. Within the social sciences, some have relatively strong grammars i.e., their conceptual syntax enables "relatively precise empirical descriptions" (e.g., linguistics, economics); while others have weak grammars (e.g., sociology, education). Bernstein describes mathematics as a horizontal knowledge structure as it "consists of a set of discrete languages for particular problems", with a strong grammar[16]. While mathematics largely does not have empirical referents, there is little dispute as to what is and is not mathematics from the point of view of the kinds of terms used, and the ways they are connected and presented. Education, in contrast, is horizontally structured but with a weak grammar. Empirical descriptions of educational phenomena vary widely, a function of the multiple languages used to describe these, many of which lack precision.

What then might be the effects of weakening of knowledge boundaries between two domains of knowledge when one has a weak and the other a strong grammar? There has been a great deal of contestation over curriculum policy in South Africa and elsewhere that has advocated weakening the boundary between mathematics and everyday knowledge. The motivation for this move lies in the view that horizontal discourses of 'realistic' or 'relevant' settings for mathematics provide access to and meaning for abstract mathematical ideas. The critique of this arises within a Bernstein framework, and posits that 'realistic' or 'relevant' settings can work instead to background mathematical principles, and so result in denying access, particularly to those already disadvantaged by dominant discourses[17]. In a similar vein, Taylor, Muller and Vinjevold (2003), for example, draw on Bernstein to argue that integration in teacher education through the weakening of boundaries around content knowledge can result in methods of teaching dominating pedagogic discourse in teacher education at the expense of content development of teachers. Here too, effects will then be skewed against the already disadvantaged. They see a danger for teachers in greatest need for access to further content knowledge being subjected to an education dominated instead by examples of supposed good practice. In QUANTUM we share this concern. However, our assumptions are different. In our analysis, forms of integration are inevitable in contemporary educational practice. The issue is how these are and can be accomplished without damage to agents. Hence the need to understand varying practices and thus our focus in QUANTUM: what is at work in mathematics teacher education in South Africa when there is more or less integration of mathematics and teaching/education in these programmes?

Putting the Evaluative Rule to Work in the Study

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[16] We are not convinced of the distinction drawn here between mathematics as horizontal and physics as vertical. Physics too has discrete languages. The inter-related distinctions here are, in our view, questionable, but full discussion beyond the scope of this chapter. The significant distinction, which in our view is productive and illuminating, is between week and strong grammars. Physics and mathematics both have strong grammars.

[17] It is beyond the scope of this chapter to detail this debate. It is well known in the field of mathematics education. Interested readers might find the work of Cooper and Dunne (1998) interesting, as well as the debate between Jo Boaler (2002) and Sarah Lubienski (2000) in the Journal for Research in Mathematics Education.

One major difficulty that arises in an integrated educational code (or when here is a weakening of classification in a curriculum), is what is to be assessed and the form of assessment[18]. Criteria must be worked out. Whether or not this is explicitly done, criteria will emerge and be transmitted. In QUANTUM's terms, implicit criteria can be rendered visible because any act of evaluation has to appeal to some or other authorising ground in order to justify the selection of criteria. In Adler and Davis (2006) and Davis, Adler and Parker (2007), we describe in more detail how we have worked from the proposition that the authorising grounds at work in teacher education pedagogic practice illuminate what comes to be privileged (in terms of knowledges and their integration)[19]. Given the complexity of teaching and more so teacher education, we started from the assumption, and related concern, that what comes to be taken as the grounds for evaluation is likely to vary substantially within and across sites of pedagogic practice in teacher education. Our methodology and language of description have allowed us to examine the diverse ways in which mathematics and teaching come to be co-produced in mathematics teacher education practice.

The QUANTUM research project began in 2003 with a survey of 11 higher education institutions offering formalised (i.e., accredited) mathematics teacher education programmes in South Africa. We collected information on courses taught including formal assessments. Phase 1 of the overall study focused on formal assessment carried out across courses in our data archive. We focused on actual assessment tasks, examining what and how mathematics and teaching competence were expected to be demonstrated in these tasks. We developed an analytic tool, using the notion of "unpacking" (Ball, Bass, & Hill, 2004), but redescribing it in line with our methodology. For Ball et al., "unpacking" captures the specificity of mathematical know-how required in the practice of teaching. We were particularly interested in whether assessment tasks demanded some form of 'unpacking' of mathematics.

A full account of this phase of QUANTUM's work is provided in Adler and Davis (2006). We started from the assumption that there are (in the main) two specialised knowledges to be (re)produced in mathematics teacher education: mathematics and mathematics teaching. That these knowledges are specialized implies there is some degree of internal coherence and consistency. However, in line with the discussion on knowledge structures and grammars above, the ways in which coherence and consistency are established in mathematics and mathematics teaching differ. In mathematics a strong internal "grammar" allows for a degree of relatively unambiguous evaluation of that which is offered as mathematical knowledge; in mathematics teaching the ambiguity is greatly increased because the field is populated by academic, professional, bureaucratic, political and even popular discourses. However, we asserted that despite those differences, where the

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[18] See Moore (2000) for an interesting discussion of the challenges of disciplinary integration in a university foundation course in South Africa, and how these manifested in assessment practices.

[19] See Davis and Johnson (2007) for further development of 'grounds' at work in school mathematics classrooms.

knowledge to be reproduced is relatively coherent and consistent, justifications can be structured in a manner that conforms to the formal features of syllogistic reasoning. Whether or not explicit coherent reasoning (be it mathematical reasoning or reasoning about teaching mathematics) was required by tasks thus provided the analytic resource we needed to identify "unpacking" consistently across different tasks.

Our examination of each task involved identifying the primary and secondary objects (mathematics and/or teaching) of the task, and then whether an understanding of the logical chains (explicit coherent reasoning) relevant to the knowledge to be reproduced was explicitly demanded. As these tasks arise in mathematics teacher education, we expected that their objects may well be both teaching and mathematics and that they could vary in their demands for unpacking. Our analysis of tasks across formal evaluations in our data set was very interesting. Simply, we found that the kind of mathematical work required in teaching was infrequently assessed, with assessment tasks in mathematics focused predominantly on the reproduction of some mathematical content or skill. There was evidence, though limited and infrequent, of assessment of 'unpacking' of mathematical ideas – that specific mathematics teachers need to know and know how to use in practice to make mathematics learnable in school. There was thus a disjuncture between what is valued at the level of intention, and what comes to count as legitimate and valued knowledge in mathematics teacher education. Of course, this analysis did not provide any insight into the pedagogical practice of which these assessments were but a part.

In phase 2 of our study, we focused in on in-depth study of selected courses. This, in turn, required an elaboration of the language we had developed so far, the details of which are in Davis, Adler and Parker (2007). Pedagogic practice functions over time, unlike static assessment tasks. The unit of analysis thus required rethinking. As already noted, we accepted as axiomatic that pedagogic practice entails continuous evaluation, the purpose of which is to transmit criteria for the production of legitimate texts. Further, any evaluative act, implicitly or explicitly, has to appeal to some or other authorising ground in order to justify the selection of criteria. Our unit of analysis became what we call an *evaluative event*, that is, a teaching-learning sequence that can be recognised as focused on the 'pedagogising' of particular mathematics and/or teaching content. In other words, an evaluative event is an evaluative sequence aimed at the constitution of a particular mathematics/teaching object.

Each course, all its contact sessions and related materials, were analysed, and chunked into evaluative events. Following on from Phase 1, after identifying starting and endpoints of each event or sub-event, we first coded whether the object of attention was mathematical and/or teaching, and then whether elements of the object(s) were the focus of study (and therefore coded as M and/or T) or were assumed background knowledge (and then coded either m or t). We worked with the idea that in pedagogic practice, in order for some content to be learned it has to be represented as an object available for semiotic mediation in pedagogic interactions between teacher and learner. The semiotic mediation that follows involves moments of pedagogic reflection that in turn involve (following Davis, 2001) pedagogic judgement. All judgement, however, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly. Legitimating appeals can be thought of as qualifying reflection in attempts to fix meaning. We therefore examined *what* was appealed to and *how* appeals were made over time and in each course, in order to deliver up insights into the constitution of MfT in mathematics teacher education.

Given the complexity of teaching, and more so of teacher education, as previously intimated, we expected that what came to be taken as the grounds for evaluation was likely to vary substantially within and across the courses we were studying. Indeed, through interaction with the data, we eventually described the grounds appealed to across the three courses in terms of six ideal-typical categories: (1) mathematics, (2) mathematics education, (3) the everyday, (4) experience of teaching, (5) the official school curriculum, and (6) the authority of the adept. In each course we found differences in what was appealed to and how, differences that point to very different opportunities for teachers to (re)learn *mathematics for teaching*.

In one course (focused on teaching and learning algebra) mathematics was integrated with methods for teaching mathematics. In this course, the grounding of objects reflected on during class sessions was predominantly in what we called empirical mathematics (particular examples). In a course that focused on teaching and learning mathematical reasoning, the emphasis was in the domain of mathematics education, and so specific mathematics backgrounded. As could then be expected, a substantial grounding of objects reflected on during class sessions in this course was in mathematics education, particularly texts reporting research related to teaching and learning mathematics. Interestingly, when mathematical objects were in focus, and this occurred through the class, grounding for these was both empirical (with examples) and principled (discussion was expected to conform to demands of mathematical discourse).

There are many reasons to explain why these two courses differed as such. Our interest, however, was that very different forms of *mathematics for teaching* were constituted in these courses, offering very different opportunities for learning. Mathematics teachers, whatever level, in our view, need to grasp mathematics in principled ways if they are themselves to enable mathematical learning in their classrooms. In-service mathematics teacher education should offer opportunities for engaging with mathematics as a principled activity. Of course, this is not to say that these two courses each capture the *mathematics for teaching* in the overall programmes of which they were each but a part. In Davis et al. (2007) our discussion of these courses is elaborated through a further analysis of how in each, the way teaching is modelled appears to link with what and how mathematics and teaching are integrated, and then too with how *mathematics for teaching* is constituted. We argue there that modelling the practice is a necessary feature of all teacher education[20]. There needs to be some demonstration/experience (real or virtual) of the valued practice; that is, of some image of what mathematics teaching performances should look like. In the Algebra course, the model was located in the performance of the lecturer whose concern (stated repeatedly through the course) was that the teachers themselves experience particular ways of learning mathematics. This experiential base was believed to be necessary if they were to enable others to learn in the same way. The mathematical examples and activities in the course thus mirrored those the teachers were to use in their Grades 7 – 9 algebra class. In the Reasoning course the model of teaching was externalised from both the lecturer and the teacher-students themselves, and located in images and records of the practice of teaching: particularly in videotapes of local teachers teaching mathematical reasoning, and related transcripts and copies of learner work. The externalising was supported by what we have called discursive resources (texts explaining, arguing, describing practice in systematic ways).

Our findings in both phases of the study need to be understood as a result of a particular lens, a lens that we believe has enabled a systematic description of what is going on ‘inside’ teacher education practice at two inter-related levels. The first level is ‘what’ comes to be the content of *mathematics for teaching*, i.e., the mathematical content and practices offered in these courses. We are calling this MfT. It is not an idealised or advocated set of contents or practices, but rather a description of ‘what’ is recognised through our gaze. Some aspects of MfT here can be seen as closer to SMK, and others to PCK in Shulman’s terms. However, each of these categories is limiting in describing ‘what’ mathematics is offered in these courses. At the second level, is the ‘how’. This content is structured by a particular pedagogic discourse; and a key component in the ‘how’ that has emerged in the study, is the projection and modelling of the activity of teaching itself. In Bernstein’s terms we have seen, through an examination of evaluation at work and of how images of teaching are projected, that different MfT is offered to teachers in these programmes. The research we have done thus suggests in addition, that developing descriptions of what does or should constitute maths for teaching outside of a conception of how teaching is modelled is only half the story[21].

MATHEMATICS FOR TEACHING: A SOCIAL PRODUCTION

We stated in the introduction to this chapter that studies and developments related to *mathematics for teaching* have their roots in Shulman’s seminal work in the 1980s that placed disciplinary knowledge at the heart of the professional knowledge base of teaching. We also noted that while there has been considerable

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[20] This further supports our assumption that forms of integration are internal to pedagogic practice in teacher education.

[21] We note here that a similar point is made in Margolinas, Coulange and Bessot (2005) pointing further to resonance between the orientation to knowledge for teaching in QUANTUM and didactical theory as developed and used in studies in France.

research, the discourse of *mathematics for teaching* is fledgling. Neither of the two studies drew directly on the categories of professional knowledge as posited by Shulman, despite being driven by the same concern: to develop or deepen mathematical knowledge as it is (or needs to be) used in teaching, and a starting point that they way mathematics is used in teaching has a specificity. From our perspective, all mathematical activity (and hence all mathematics wherever it is learned) is directed towards some purpose, and within teacher education, this would be for mathematics teaching, and/or becoming a mathematics teacher. While the notion of PCK in particular is compelling in teaching and teacher education – it emphasises that pedagogic reasoning in mathematics teaching is content-filled – it does not live outside of the institutions where it functions – and these are inherently social. There are limits to the appropriateness of general categories like PCK and SMK, as well as to the distinctions between them.

Shulman's work spurred several studies (and continues to do so) attempting to build on his notions, particularly PCK, but as Ball, Thames and Phelps (2007) point out, these notions remain poorly defined. In this paper Ball et al. pull together the accumulation of their work over the past decade that has included (a) describing *mathematics for teaching* from close observation of a detailed archive of a year of mathematics teaching in a third-grade class taught by Ball in the United States and (b) developing measures of content knowledge for teaching. This research has led them to strengthen and elaborate Shulman's initial work by providing clear definitions and examplars of distinctive categories within and across SMK and PCK. A particular move they make is to define two new categories within SMK – or content knowledge for teaching: what they call common content knowledge and specialised content knowledge. They argue that this distinction is necessary to capture the specificity of teachers' mathematical work – and that recognition of this specificity lies at the heart of effective mathematics teaching. In simple terms, teachers need to know aspects of mathematics that is not required by 'others' (i.e., in common use). But what is common use? From a social epistemological perspective, all mathematical activity is towards some purpose, and occurs within some or other (social) institution. The notion of 'common' content knowledge is thus problematic, and so too then, the marking out of specialised content knowledge.

We nevertheless share with Ball and her colleagues a concern with *mathematics for teaching*. In the two studies we have presented here, we have shown how two different social epistemologies have been productive for studying *mathematics for teaching*. The Mozambique study described and explained teachers' evolving personal relations to the limit through their participation and engagement with this concept in a new institution. We showed that this evolution was uneven, across teachers, and then also in relation to different aspects of *mathematics for teaching*. We elaborated in particular, the relatively poor evolution of the teachers' grasp of graphical representations and the ε-δ definition. We argued that these outcomes were a function of the strength of the research institution relative to the dominant institutions of the secondary school and Pedagogical University. An important element of this argument was an interpretation of aspects of *mathematics for teaching* through an ATD lens that reflected the limits of the distinction posited by Shulman between SMK and PCK. The evolution of each of the teachers' personal relation to the limit concept was described in terms of both subject knowledge and knowledge of teaching. Emerging from this study is the observation that professional development programmes for practicing teachers in a context like Mozambique need to provide opportunities for substantive engagement with the content of mathematics, opportunities that were not available in the research institution set up, despite placing mathematics at the centre. So where and how then, is this engagement with mathematics to function in mathematics teacher education?

In the South African study of courses within formalised in-service mathematics teacher education programmes – where engagement with mathematics was a goal - we described the mathematics that came to be constituted (came to 'live') in and across different courses. By examining evaluation at work in the courses we were able to 'see' what and how mathematics and teaching are co-constituted through pedagogic discourse. We showed that different models of teaching combine with varying selections from mathematics, mathematics education and teaching practice to produce different kinds of opportunities for teachers in these courses to learn mathematics (for teaching).

Separately and together these two studies demonstrate that *mathematics for teaching*, and its learning in any institutional setting can only be grasped through a language that positions *mathematics for teaching* as structured by, and structuring of, the pedagogic discourse (in Bernstein's terms) or the institution (in Chevallard's terms) in which it 'lives'. Both provide strong conceptual tools with which to interrogate how mathematics is recontextualised in pedagogic settings. Separately and together they contribute to the growing body of knowledge related to the *what* of mathematics teacher education, and particularly to subject knowledge for teaching.

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APPENDIX 1

Categories of teacher's mathematical knowledge about the graphical register (reading and sketching)

GRR1	The teacher is not able to read any limit from the graphs.
GRR2	The teacher is able to read some limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote).
GRR3	The teacher is able to read limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote), and infinite limits at infinity ($x\to \infty$, $y\to \infty$).
GRR4	The teacher is able to read limits along a vertical or a horizontal asymptote (even when the graph crosses the asymptote), and infinite limits at infinity ($x\to \infty$, $y\to \infty$).
GRR5	The teacher is able to read most limits but faces small difficulties.
GRR6	The teacher is able to read all kinds of limits.
GRS1	The teacher is not able to sketch any graph using limits or asymptotes.
GRS2	The teacher is not able to indicate any limit on axes. He is able to sketch a standard graph having two asymptotes, one vertical and one horizontal.
GRS3	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He does not acknowledge that drawing several branches may produce a graph that is not a function.
GRS4	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He acknowledges that the produced graph does not represent a function.
GRS5	The teacher indicates limits along a vertical or a horizontal asymptote as a local behaviour.
GRS6	The teacher is able to indicate any kind of limit on axes.

Categories of teacher's ideas about the use of graphs to teach limits

GR-T1	The teacher would not use graphs when teaching limits.
GR-T2	The teacher acknowledges the importance of the graphical register in teaching limits.
GR-T3	The teacher acknowledges the importance of the graphical register and explains how he would use it or articulate it with other registers.