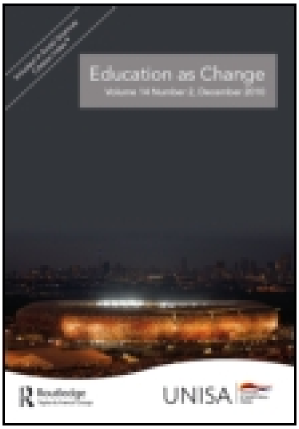

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Teachers as researchers: Placing mathematics at the core

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Teachers as researchers: Placing mathematics at the core

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Abstract

Teachers involved in action research in mathematics classrooms typically focus on their pedagogical practice, and rarely challenge the mathematical content of their teaching. The first part of this article supports this claim in terms of an analysis of the teachers-as-researchers movement. It then reports on a study where teachers were researching aspects of mathematics for the teaching of limits of functions, in which mathematical and pedagogical issues were intertwined. The aim of the study was to analyse how teachers' mathematical knowledge evolves through their participation in a research project. The study showed that whilst their knowledge evolved for some aspects of limits during the research process this was not the case in two crucial aspects of this knowledge: the ε-δ definition and the graphical representation of limits. Furthermore, the study revealed the discomfort of an experienced teacher when facing challenges to his own mathematical knowledge and therefore the content of his teaching.

Key words: mathematics education, teachers as researchers, mathematics for teaching

Introduction

At the core of the teacher-as-researchers movement is the expectation that teachers researching their own practices in classrooms 'learn more about student understanding of mathematics, mathematics itself, and themselves as teachers' (D' Ambrosio 1998:146). In this article, we show how these expectations, in particular with respect to 'mathematics itself', may not be met when the mathematics is not explicitly at the centre of the research. This is an important consideration for teachers researching their own practice in the Mozambican context, where the focus in mathematics teaching has traditionally been on the execution of algorithms rather than on applications or mathematical reasoning. In this context most teachers do not criticise syllabi and textbooks (designed by teachers from Mozambican institutions) and the same teaching practices and pedagogies are perpetuated in schools, textbooks and syllabi year after year. Involving teachers in research is a potential way of breaking the cycle: hopefully, this will make teachers reflect on their content knowledge, the content of their teaching and its consequences for students in terms of mathematical knowledge and poor reasoning practices. The article examines this issue and offers two inter-related arguments with respect to mathematics teachers researching their own practice:

The first argument is that there are limitations as to what is possible for teachers to learn mathematically, through researching aspects of their practice. Secondly, practising teachers face considerable discomfort when faced with challenges pertaining to their own mathematical knowledge and so to the content

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of their teaching. These arguments are primarily drawn from a study of the evolution of Mozambican teachers' knowledge of the limit concept through their participation in a research group.

We begin the article with an analysis of the teachers-as-researchers movement in general and in mathematics education in particular. Through this analysis we build support for the claim that there is a pedagogical emphasis in much of this research. We then describe a study of the evolution of teachers' knowledge of limits of functions through their participation in a research project. The description is framed by the Anthropological Theory of Didactics (ATD). The study thus has particular mathematical content, together with its learning by teachers, as its focus; it deliberately brings mathematics into the centre of the teachers' research. We use our findings to reflect back on the teacher-as-researchers movement, particularly with respect to teachers learning mathematics itself, and so for teaching.

I. The teachers-as-researchers movement

According to Elliott (1991), the teachers-as-researchers movement emerged in England during the 1960s, in the context of curriculum reform. Initially, it focused on the teaching of humanities, with teachers working together in cross-subject teams on integrated curricula. The research occurred as a response to particular questions and issues as they arose. Its focus was on the improvement of practice rather than the production of knowledge.

This movement extended in the 1980s into what is usually known as *the teacher research movement*. In a paper titled 'The Teacher Research Movement: A Decade Later', Cochran-Smith and Lytle (1999) review papers and books published in the United States and in England in the 1980s which disseminate some experiences of teacher research. They argue that

> the visions of educational research embedded in these writings shared a grounding in critical and democratic social theory and in explicit rejection of the authority of professional experts who produced and accumulated knowledge in 'scientific' research settings for use of others in practical settings (1999:16).

The positioning of teachers as central to educational research focusing on improving practice appears to be the main feature of the teacher research movement. Within this movement, teachers are no longer considered as mere consumers of knowledge produced by experts, but as producers and mediators of knowledge, even if it is local knowledge, to be used in a specific school or classroom. In fact, in much of their research, teachers focus on their own classroom practice. With an eye focused on the practice of teaching, it appears that most of these research studies background the academic content, weakening the link between pedagogical practices and their particular contents.

The teacher research movement evoked numerous debates, but the relationship between specific contents being taught and pedagogic strategies studies were not in focus. Contention pivoted around whether the outputs of such studies could be regarded as research. Many research endeavours conducted by teachers do not fill the requisites of formal research, such as systematic collection and analysis of data, as well as dissemination of the research results (Richardson 1994; Cochran-Smith & Lytle 1990 and 1999; Breen 2003). Richardson (1994) argues that teacher research is a 'confusing concept' (1994:6), as there are several motivations captured in this notion. She distinguishes two forms of teacher research in practice: practical inquiry and formal research. For Richardson, several approaches such as *teacher as reflective practitioner* and *action research* should be qualified as practical enquiry. This kind of research does not aim to produce general results concerning educational practice, but rather suggests new ways of looking at the context and possibilities for changes in practice. It produces local knowledge for the purpose of improving in one's everyday life and is not generally disseminated. Formal research means to contribute to the knowledge of a larger community.

Within this debate, others (e.g. Brown 1997) only use the term *action research* to refer to a more systematic, self-critical enquiry that has been made public. According to Crawford and Adler (1996) the term *action research* is widely used to describe investigations and inquiry undertaken with an intent to change professional practice or social institutions through the active and transformative participation of those working within a particular setting in the research process. A major aim of most action research projects is the generation of knowledge among people in organisational or institutional settings that is actionable – that is, research that can be used as a basis for conscious action (ibid:1187). This generative research has usually been conducted by teams involving teachers and educational researchers. Indeed, two kinds of research can be distinguished: formal research which aims to contribute to the larger mathematics education community's knowledge, and less formal research usually done by teachers and which aims to produce local knowledge and improve teachers' practice. An interesting example of the former is the COREM project, reported in Novotná (2003), where collaboration included elementary school teachers, school psychologists, students of didactics, mathematicians and researchers who were university teacher trainers. Goals were explicit with respect to knowledge generation – including the advancement of knowledge of the mathematics education phenomena, knowledge of situations that enabled improved learning, and knowledge related to possibilities for related teacher education.

This article, and the study it reports, is concerned with the second kind of research, specifically in mathematics education.

Teachers as researchers in mathematics education

In mathematics education worldwide, the teacher-as-researcher movement has become an important part of many teacher education programmes. It also has been the subject of debate within the mathematics educators' community and of several papers presenting the results of these programmes or discussing certain aspects of teacher research. We discuss a wide-ranging selection from this research, which, as will be seen, focuses mainly on teachers' practices.

In 1988, the International Group for the Psychology of Mathematics Education (PME) started a working group called 'Teachers as Researchers'. This group met annually for nine years and published a book based on contributions from its members (Zack, Mousley & Breen 1997). The book comprised accounts of teachers' different experiences of enquiry in several countries and using several methods; the aim of these enquiries was basically to improve practice. During this time, and also within the context of PME, Adler (1992) reports a case study of a middle-class mathematics teacher researching his interactions with learners and their interaction with each other during his postgraduate studies. Through this research, he realised that he dominated classroom interaction and that his mediation was gendered. Here, the mathematics being taught was clearly out of focus.

D'Ambrosio (1998), on the other hand, relates two experiences of learning through teacher research. In the first one, pre-service secondary mathematics teachers formed a research team which investigated children's understanding of fractions. D'Ambrosio reports an increase in students' reflective thinking about children's perspectives on fractions and their constructions around fractions. Such reflections would not have been generated by teaching experience alone. In the second experience, teachers were encouraged to identify a research question related to their classroom practice through personal journals. A certain pattern emerged in the teachers' choice: several of them chose to look at how to manage their classroom better; others chose to study a student or a small group of students and their learning. Every week the teachers presented their findings to their small working group. D'Ambrosio concluded that 'the teachers who engaged in teacher research found themselves questioning their practice, as well as wondering and planning what they might do differently' (1998:155). It is interesting to note here that while 'fractions' was the topic being taught, this content was assumed rather than interrogated as part of the reflection and research. Similarly, Edwards and Hensien (1999) describe action research collaboration between a middle-school mathematics teacher and a mathematics teacher educator which involved observation and discussion of lessons and exchange of roles in the classroom. The analysis of the teacher's narrative of this collaboration as well as the teacher's regular reflections on her beliefs and practices were important to her process of change.

Some projects report outcomes beyond the local practice of teachers. Hatch and Shiu (1998), for example, report a case study of a primary school teacher in an in-service course researching her own practice through the analysis of a class transcript and a reflective journal. They argue that she contributed not only to developing knowledge of her own practice but potentially to the accumulated knowledge of the research community. Halai (1999) reports on action research conducted by mathematics teachers in Karachi involving university researchers as facilitators. The teachers and university researchers used participant observation, field notes and reflective journals. It is nevertheless fair to say, again, that while these moved beyond the local, pedagogic practice was in the foreground in these studies, with the mathematics taught taken for granted and so backgrounded.

Jaworski (1998) identified the backgrounding of mathematical issues in her MTE (Mathematics Teacher Enquiry) project. This project involved six secondary mathematics teachers undertaking their own research independently of an academic programme. The teachers were invited to identify a question they were interested in researching. Jaworski points out that, during this research, the teachers focused their attention on pedagogical issues rather than on mathematical issues. Decisions about 'what mathematics should be done, what classroom tasks would be appropriate, and what outcomes would be desired' while a normal part of the teaching process, were hard to extract as 'problematically related to the research issues' (1998:25).

She asks the question 'How might mathematics issues become more overt in the research project?' (1998:29).

In fact, as is evident in these summative points, in most of the papers presented above, the focus is on teachers' classroom practices. Out of focus, treated as unproblematic, and hence split off it seems, is attention to the knowledge to be taught. This is the case even where the teachers investigated a specific content. In D'Ambrosio's study (1998), for example, the mathematical domain is fractions, but the investigations focus on students' difficulties. Challenging the content of teaching was not the aim of the project. According to D'Ambrosio, 'teacher research was used as a tool for developing, encouraging, and sustaining teachers' reflective practice' (1998:155). Mathematical content was not a focus.

In all these projects it appears that the mathematical content to be taught, or more specifically in the language of the study as reported below, the content itself, and the institutional relationships to such content, are taken for granted. Teachers are not engaged in systematic study of the content itself, and so while they may challenge their own teaching practices, they do not specifically engage with the mathematical content or dispute what is being taught– that is, its institutional forms.

There are, however, other reports of studies that mention some change, or some possible change, in teachers' knowledge of mathematics– i.e., studies in which the content of teaching is more or less in focus. Mousley (1992), for example, reports the results of a year course in an off-campus mode called Mathematics Curricula. Course participants used a cycle of action research in a chosen area. They were required to work with colleagues. A representative sample of 60 teachers was then contacted by mail, telephone or a personal interview, and asked about the impact of the course. It was found that there was not only some ongoing restructuring of pedagogy in terms of content, organisation and classroom interaction, but also growth of understanding about (1) the nature of mathematics, (2) the processes of teaching and learning of mathematics, (3) the power of institutional contexts of teaching and learning, and (4) the processes of pedagogical change (Mousley 1992:138).

Although the aim of Mousley's project was to improve practice, it also shows that through their research, teachers' knowledge of mathematics evolved and they also became aware of the weight of institutional constraints on this knowledge. The notion of mathematics as a stable body of knowledge and skills to be transmitted and practised became problematic. Questioning traditional classroom practices provided an incentive for teachers to confront given curriculum content (1992:139). Mousley concluded that participatory, experience-based research has the power to emancipate some teachers from taken-for-granted classroom routines, which constrain and control mathematical learning. The dialectical interaction of reflection combined with social interaction allowed innovation in the nature of teaching and learning mathematics as well as in curriculum content (1992:143).

This experience shows that through research and interaction teachers *can be* led to challenge institutional relations to mathematics.

In the first edition of the *International Handbook of Mathematics Education*, Crawford and Adler suggest that:

> It seems possible if teachers and student-teachers act in generative, research-like ways, they may learn about the teaching/learning process, and about mathematics, in ways that empower them to better meet the needs of their students (1996:1187)

These authors seem to avoid the distinction between practical inquiry and more formal research, using the term 'research-like ways'. The focus is on teachers' personal learning by researching, not only their own practice, but also mathematics. They argue, as does Ball (1988), that since the quality of teachers' mathematical knowledge is strongly influenced by their own experience as students, they need to unlearn the old conceptions of mathematics derived from their schooling experience. The experiences of 'teachers' voices' in South Africa and of a program of action-research with student teachers in Australia led Crawford and Adler (1996) to conclude that research helps teachers to challenge their practice and their conception of mathematics. Unlike Ball, however, who focuses on what prospective teachers of mathematics know (i.e., knowledge *of* mathematics), *and* what they believe and care about – mathematics and its teaching and learning – what they need to (re)learn in teacher education, Crawford and Adler's discussion of prospective teacher learning through research remains focused only on teachers' 'views of mathematics, which includes capabilities for problem posing ..., investigation inquiry ...', (i.e., knowledge *about* mathematics) none of which are elaborated mathematically in the paper (1996:1200–1201).

Another research project reporting changes in teachers' knowledge *of* mathematics is the PLESME project (Graven 2005) – in this project mathematical knowledge and mathematics pedagogical knowledge were intertwined. PLESME focused on the development of mathematical meaning and pedagogical forms simultaneously (2005:219). Using this two-year INSET project as an empirical field for her research, Graven investigated the nature of mathematics teachers' learning within a community of practice (2005:207). She argues that most of the literature on teacher development indicates a focus on teacher change. In the South African context, the curriculum support materials call 'for radical teacher change where old practice is completely replaced by new practice'. This view of teacher change is disempowering for teachers (2005:223). On the other hand, the PLESME programme was based on a conception of learning as a life-long process, expecting teachers to build their own knowledge.

In the wider study from which her paper was drawn (Graven 2002), and through her examination of changing identity and practice across ten teachers in the study, Graven shows that growth in mathematical knowledge used in teaching was uneven. Some teachers' mathematical horizon was limited, and this continued to constrain their teaching, as well as prevent them from extending their limited knowledge through a focus on teaching practice. In contrast, there were more decisive mathematically coherent shifts in teachers whose mathematical base was stronger.

Within the research forum 'Teachers researching with university academics' at PME30, several teachers and researchers report different ways of collaboration between teachers and researchers, most of which are aimed at improving teachers' practice (Novotná, Zack, Rosen, Lebethe, Brown & Breen 2006). However, some reports also refer to the mathematical content and to the teachers' mathematical knowledge.

For example, Rosen (2006) states: 'I found conventional ways of doing mathematics as prescribed in official textbooks were not working for me in my classroom' (2006:113). He then tried to

> share with other teachers, especially those who are challenging school situations in order to share with them what has worked for me, and to help them explore their own ways of doing mathematics both for themselves and with the children (2006:113).

This teacher is challenging the institutional relationship to mathematics as presented in textbooks. He elicits support from other colleagues because it is not easy to be a 'bad subject' of the institution.

Within the same Research Forum, Hospesová, Machacková and Tichá (2006) explain:

> At the beginning of our work on the mathematical topic of the teaching experiment, we discussed (and when necessary the researchers summarised for the teachers) its mathematical background and its possible didactic elaboration (2006:101).

They conclude that the teachers' 'low self-evaluation and uncertainty in their own mathematical understanding may be impeding their progress' (2006:101). In another report (Zack & Reid 2006), an elementary school teacher explains the role of the researcher as follows:

> I have enlisted David's help on a number of occasions when aspects in the mathematics have puzzled and intrigued me. My background in mathematics is weak. At times I feel vulnerable when I do not understand, and I will only seek help if I feel I can trust the other person to not make me feel inept' (2006:117).

These latter two reports demonstrate the weakness of some teachers' mathematical knowledge and the discomfort of challenging this knowledge. We will see that an experienced teacher felt the same discomfort when confronted with his mathematical mistakes in the research presented in this article.

This non-exhaustive review of papers about the teachers-as-researchers movement shows the range of experiences in this domain in terms of research topics and methodology. However, some common trends are identifiable across these reports.

Firstly, they seem to share a common conception of the teacher as a producer of knowledge and not as a mere consumer of knowledge produced by other individuals, particularly academics. Secondly, in most of these research projects teachers worked together in groups, the research team being composed of either pre-service or in-service teachers. Interaction between teachers, or between teachers and mathematics educators, allowed them to deepen the analysis of their practices and of their difficulties. Finally, in all projects discussed above, teachers investigated some aspect of their own teaching, or some problem of student learning. It seems that, particularly when asked to choose a research topic, teachers question their own teaching, or their students' performance and difficulties, but take for granted the content usually taught within the institution.

Of interest and concern, then, are the possibilities for mathematical learning in a teacher research project with some mathematical content at its centre. In the study presented in the next section, teachers were not researching their own practice but the *Mathematics for Teaching* (MfT) limits of functions for secondary school level. The notion of MfT limits of functions has been described in detail in previous articles (see for example, Huillet 2009). It is briefly elaborated within the following discussion of the research project.

II. The research project

The research project mentioned in this article aimed to investigate how secondary school mathematics teachers' knowledge of limits of functions developed through their participation in a research group. The limit concept was chosen because it is the first higher mathematics concept met by students in secondary schools. It is a very abstract concept and students usually experience many difficulties when learning it in schools, and even at university. For these reasons mathematics educators across a range of countries have focused their research on different aspects of the learning of this concept.

The study, in terms of its design and analytic frame, was informed by the anthropological theory of didactics (ATD). After a brief introduction to this framework, we explain how the research was designed and present some findings that show that whilst teachers' knowledge evolved for some aspects of limits during the research process, this did not happen for two crucial aspects of this knowledge: the ε-δ definition and the graphical representation of limits. Furthermore, we illuminate the discomfort of an experienced teacher as he confronted his own mathematical knowledge and so too the content of his teaching. Implications follow for mathematics teacher research and its varied intentions.

The anthropological theory of didactics

ATD locates mathematical activity as well as the activity of studying mathematics within a set of human activities and social institutions (Chevallard 1992). It considers that 'everything is an object' and that an object exists if at least one person or institution relates to this object. To each institution is associated a set of 'institutional objects' for which an institutional relationship, with stable elements, is established.

An individual establishes a personal relationship to some object of knowledge if s/he has been in contact with one or several institutions where this object of knowledge is found. S/he is a 'good' subject of an institution relative to some object of knowledge if his/her personal relationship to this object is judged to be consistent with the institutional relationship (Chevallard 1992).

In Mozambique (as elsewhere), the relationship that school mathematics' teachers establish with the limit concept is shaped by the relationship of institutions where the teachers have been exposed to this concept. For most teachers this contact has occurred in Mozambican institutions (in secondary school as students, in university as students, and in secondary school as teachers). The researcher used ATD in the first place as a tool for analysing the institutional relationship of these Mozambican didactic institutions to the limit concept, with particular focus on the secondary school institution and the Pedagogical University (PU), where most mathematics teachers are trained. For each of these institutions, the institutional relationship to limits of functions was analysed through the examination of the syllabus, the national examinations (secondary school), worksheets used in secondary schools (there were no textbook for this level in Mozambique at the beginning of the project), textbooks used at the PU and the exercise book of a PU student. This analysis highlighted a dichotomy between two components: the algebra of limits based on the ε-δ definition, and the existence of limits, based on algebraic transformations to evaluate limits. This dichotomy, which also exists in other countries' secondary school curricula (Barbé, Bosch, Espinoza & Gascón 2005) and has been explained by the nature of the limit concept, seems to be exacerbated in the Mozambican case. This may explain the limited personal relationship to limits of Mozambican teachers (see Huillet 2007).

According to ATD, these institutional relationships strongly shape teachers' personal relationships to the limit concept, and this personal relationship constrains any challenge to institutional routines. Moreover, this personal relationship can only evolve if teachers are in contact with this concept through a new institution (or new institutions) where this concept is not limited to these two components but lives in a more elaborated way. For example, numerical and graphical representations of limits may provide a new perspective, not usually met in regular Mozambican institutions.

Obviously, changing one's personal relationship to limits does not automatically result in changing the way limits are taught in schools. Other institutional or personal constraints may influence teachers' ways of teaching. However, the evolution of their knowledge is a necessary, although not sufficient, condition for any change in the way of teaching limits in Mozambican secondary schools. The methodology of this research has been designed according to these premises, and the idea that teachers could deepen their mathematical knowledge through participation in a research project.

The research design

The study involved four teachers (Abel, David, Frederico and Mateus)[1]. Each teacher was researching one aspect of limits of functions for their Bachelor or Masters Degree dissertation, according to research criteria set for those respective degrees. Abel was an experienced teacher who had taught limits in schools for many years. He was trained as a teacher in an Upper Pedagogical School in Germany, and was taking a Masters Degree in Mathematics Education in Mozambique. The other teachers were in their 5th year of teacher training at Pedagogical University in Maputo. Frederico, had taught limits for one year in a very intuitive way, without referring to its formal definition, in a professional school. The other two teachers, David and Mateus, had never taught limits because they were teaching mathematics in lower grades.

The research topics for teachers were chosen according to a previous analysis of MfT limits of functions. This analysis was performed within the ATD framework. The first author has argued elsewhere (Huillet 2009) that this analysis led to the conclusion that the usual distinction between 'purely mathematical knowledge' and mathematical knowledge used in teaching was not appropriate, because mathematics does not live in a vacuum but inside institutions. Drawing on previous research on subject matter knowledge for teaching mathematics (e.g., Even 1990), and re-interpreting this through the lens of ATD, the first author described MfT limits of functions as including the following aspects: (i) Scholarly mathematical knowledge on the concept; (ii) Knowledge about the social justification to teach this concept; (iii) Knowledge on how to organise the students' first encounter with the concept; (iv) Knowledge on the practical block (tasks and techniques); (v) Knowledge about students' conceptions and difficulties when studying this concept. In each of these components mathematical and pedagogical knowledge are intertwined.

Each teacher chose a specific topic from the list that their supervisor (the first author) provided. This list comprised the history of the limit concept and its implications for teaching (Frederico), the use of different registers in teaching the limit concept (Mateus), alternative ways of organising the first encounter with the limit concept (Abel), applications of limits of functions in mathematics and in other sciences (David).

All topics were the content of discussion seminars, where teachers shared the progress of their studies with the other teachers in the research group. Thus, all four teachers had their own agenda for participating in the research group, and all the teachers had opportunities to engage, at different levels of depth, in the different aspects of MfT the limit of function described above. The first author was the supervisor (or co-supervisor) of the teachers' research and also the facilitator of the seminars.

The new institution set up for the study included several components – individual research, supervision sessions and discussion group. Within this institution teachers could establish new connections to the limit concept; and in turn, this should allow their personal relationship to this concept to evolve. In the discussion seminars the teachers shared the progress of their research, their difficulties and their findings. Alternatively, they discussed some important topics related to limits of functions (ε-δ definition at Seminar 3; Different settings and registers at Seminar 5). The teachers were also interviewed individually by the first author at the beginning, in the middle and at the end of the research project.

Data collection and analysis

All interviews and seminars were audio-taped and transcribed. In order to explore the evolution of teachers' knowledge on limits of functions, data analysis focused on five main aspects of MfT limits: how to organise students' first encounter with this concept, the social justification for teaching limits in schools, the essential features of the limit concept (part of the scholarly mathematical knowledge), the graphical register (from the practical block), and the ε-δ definition (also from the scholarly mathematical knowledge).

Categories were defined to classify teachers' mathematical knowledge on each of these aspects and to classify their ideas about including these aspects in their further teaching of limits. The categories emerged through interaction with data analysis, and we provide details of the categories as they were defined for the graphical register (Tables 1 and 2). We have selected this aspect to exemplify the categorising, as the evolution of the teachers' knowledge of the graphical register in the study is central to our argument in the article. Teachers' initial relationships to limits, in terms of their knowledge within the five aspects of limits objects of the study, and their ideas about using these aspects in teaching, were classified using their answers to the first and second interviews and their participation in the first seminars. Their final relationships to limits for the same five aspects were classified using their answers in the third interview and their participation in the last seminars. This enabled the analysis of the evolution of each teacher's relationship to limits, and the comparison between teachers.

Table 1: Categories of teacher's mathematical knowledge about the graphical register

GRR1	The teacher is not able to read any limit from the graphs
GRR2	The teacher is able to read some limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote)
GRR3	The teacher is able to read limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote), and infinite limits at infinity ($x\to\infty, y\to\infty$)
GRR4	The teacher is able to read limits along a vertical or a horizontal asymptote (even when the graph crosses the asymptote), and infinite limits at infinity ($x\to\infty, y\to\infty$)
GRR5	The teacher is able to read most limits but faces small difficulties
GRR6	The teacher is able to read all kinds of limits
GRS1	The teacher is not able to sketch any graph using limits or asymptotes
GRS2	The teacher is not able to indicate any limit on axes. He is able to sketch a standard graph having two asymptotes, one vertical and one horizontal
GRS3	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He does not acknowledge that drawing several branches may produce a graph that is not a function
GRS4	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He acknowledges that the produced graph does not represent a function
GRS5	The teacher indicates limits along a vertical or a horizontal asymptote as a local behaviour
GRS6	The teacher is able to indicate any kind of limit on axes

Table 2: Categories of teacher's ideas about the use of graphs to teach limits

GR-T1	The teacher would not use graphs when teaching limits
GR-T2	The teacher acknowledges the importance of the graphical register in teaching limits
GR-T3	The teacher acknowledges the importance of the graphical register and explains how he would use it or articulate it with other registers

Findings

Data analysis for the five aspects mentioned above indicate that teachers' mathematical knowledge evolved substantially for the first three aspects: essential features of the concept, the social justification for teaching limits in schools, and the first encounter with the limit concept. In other words, all four teachers' knowledge of what are productive initial experiences for students when starting to learn about limits evolved over the duration of the project. Similarly, in the interviews and seminars all students displayed evolving knowledge of where and how limits fit into the curriculum and their significance for a school mathematics curriculum, together with some meta-knowledge of the limit concept (Huillet 2007). However, difficulties remained for the two last aspects, the graphical register and the ε-δ definition (Huillet 2007), aspects that were distinctly mathematical in themselves. We elaborate on the results concerning the graphical register below.

During the interviews, tasks were presented to the teachers; some of them involving reading limits from graphs and others requiring the sketching of possible graphs of functions given several limits. The same tasks were used during the third interview in order to compare teachers' answers at the beginning and at the end of the research process. In both interviews, these tasks were presented to the teachers as possible tasks for secondary school students and the teachers' opinions about these tasks were elicited through questions such as:

– Do you think that these tasks could be used in secondary schools?

– Which task would be more difficult for students?

– Do you think these tasks could help students better understand the limit concept?

Analysis of all teachers' solutions of the tasks show that they made substantial progress in reading limits from the graph of a function. However, their progress in sketching graphs using the limits of the function was not as positive. The main difficulty faced by the teachers was that they used the limit to determine a whole branch of the graph, instead of local behaviour. For example, in the first interview, David sketched the following graph in response to a question which asked the student to graphically represent a function with the following limits:

$$\lim_{x\to-\infty} f(x)=1^{+},\ \lim_{x\to+\infty} f(x)=1^{-},\ \lim_{x\to 2^{-}} f(x)=+\infty,.\ ,.\ \lim_{x\to 2^{+}} f(x)=-\infty.$$

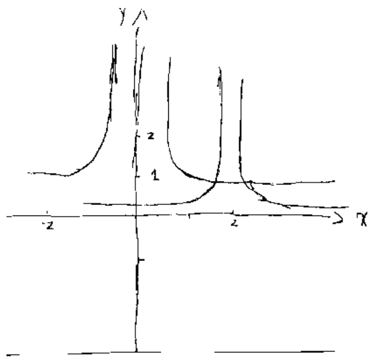

Figure 1: David's first graph

This graph does not even represent a function. David, however, was not able to see that, and this shows that he had a poor understanding of the concept of function itself.

In his last interview, David seemed to enjoy using limits to generate the graph of a function and tried to solve three such tasks. In these tasks he was able to correctly indicate several limits on a graph. For example, Figure 2 shows the graph where David represented the following limits:

$$\lim_{x\to-\infty} f(x) = 1^-, \qquad \lim_{x\to+\infty} f(x) = 1^+, \quad \lim_{x\to2^-} f(x) = -\infty. \qquad \lim_{x\to2^+} f(x) = -\infty.$$

In this graph, David correctly indicated each limit as a local behaviour of the function. However he was not able to link these limits to sketch a whole graph of a function, because he still had the misconception that a graph cannot cross any of its asymptotes.

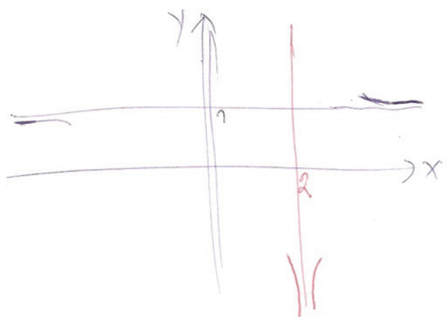

Figure 2: David's final graph

David was the one who learnt most about the graphical register during the research process. This learning of how to use a limit to determine local behaviour of the function did not happen with the other teachers. For example, during the third interview, Abel was not able to indicate any limits on a graph and said that he had never done it before. Both Frederico and Mateus drew a whole branch for a single limit, obtaining a graph similar to David's first one (Figure 1).

Two main reasons may be suggested to explain these differences. As indicated, David's knowledge of graphs evolved during the two interviews. He always tried to solve more tasks, using the supervisor's explanations when he had failed to read or use one of the limits correctly. Probably related to this was the fact that the use of limits to sketch a graph was part of his dissertation. As a consequence, he had to learn more about this activity, and he received direct feedback during the supervision sessions and during the seminars on this topic. What distinguishes David's activity in the research project from the other teachers with regard to using limits to sketch graphs is that, here specifically, his interaction with the supervisor involved direct teaching by the supervisor – and as we have discussed above, this kind of activity was not present in the other institutional forms of the research project.

The difficulties faced by teachers in understanding the ε-δ definition have been explained elsewhere by the inversion of the order of ε and δ within this definition, and by the fact that it is not used to determine the limit of the function, but rather to prove that a certain value is actually the limit of the function (Huillet 2007). In the case of the ε-δ definition, this was not a focus of the research project for any of the teachers, and so none of the teachers had an opportunity for being directly taught. While difficulties with the definition emerged in seminars, direct teaching related to these was not part of the research institution.

The difficulties with the graphical representation of functions and the ε-δ definition have been further explained by the teachers' lack of understanding of crucial mathematical concepts, such as the concept of function, and the poor status usually given to graphical representations in Mozambican didactical institutions. This does not apply to the first three aspects, which only involved general mathematical knowledge. In that case, reading books and mathematics education papers and discussing their findings within the research group seemed to allow teachers to reflect on these issues and to make links between the limit concept and other mathematical concepts.

Furthermore, the Grade 12 experienced teacher, Abel, faced other difficulties during the whole research process. In fact, this teacher was in a less comfortable position than his colleagues. While the other three teachers were researching and challenging the institutional relationship of Mozambican secondary school to limits of functions, Abel was also researching and challenging his own practice. We explained elsewhere how he realised during the research project that he had been teaching the ε-δ definition incorrectly, changing the roles of ε and δ (Huillet 2007). At another point in the research project, he also realised that he had taught L'Hôpital's Rule before teaching derivatives; consequently students would not be able to understand it. This is illustrated by these extracts from his 3rd interview.

> *I remember that, well I gave tasks about limits, er … mainly, they were polynomial functions I think, well, for me, the practical way was, you know, use what we usually call L'Hôpital's Rule, because it was practical and [sighing] but … after all, now I get to know that, well, how could I use that L'Hôpital's Rule if the students did not learn derivatives? And limits come before derivatives … But … I saw that after all I was doing a mistake by that time … but … (…) The problem is that, for me ... I saw that it was so practical, quick, hum, a quick process, then … but after all I was doing a mistake!*

Abel then explained how he introduced limits to his students.

> *I gave the definition, ok, I gave the rules, we go to the tasks. (…) Well, I was myself reduced to ... to that knowledge, thus, it's how I learnt and it's also what the textbook shows, and I'm going to pass it on [to students].*

As we see in his discourse, Abel's mathematical knowledge did not enable him to teach in a different way. His experience through the research was to come face to face with mathematical problems in his teaching that he could now identify, but could not address. It was also very hard for him to realise that he had taught in a way students could not understand, that he had made ‘a mistake’, and could only ‘pass on’ his own ‘reduced’ knowledge to his students.

Abel provides an example of why it might be that teachers who are researching their own practice prefer to look at pedagogical issues or students’ difficulties. A secondary school mathematics teacher, who has a degree in mathematics, is supposed to have strong mathematical knowledge of the topics taught in secondary school. Becoming aware of the limitations of this knowledge is disconcerting, unless it is a new topic in the curriculum. Calculus is not a new topic, and has been taught in secondary schools and universities for a long time. Qualified practising teachers, or final-year university students are supposed to ‘know’ limits of functions. They usually learn these within particular institutional requirements, where the focus is on determining limits through algebraic techniques. This approach does not necessarily facilitate deep understanding of the concept, and so teachers teach the concept according to those same institutional routines. They are not able to challenge their own personal relationship to mathematics and the consequences of this for student learning. As we have argued, they need to be in contact with this concept through an institution where it lives in a different way so as to access other features of this concept. We are arguing here, that investigating their own practice is not likely to challenge the institutional relationship and, therefore, their personal relationship to the concept. We have already mentioned that the same discomfort had been expressed by another teacher (Zack & Reid 2006).

These results highlight some of the limitations of teachers learning mathematics through participation in a research project, as well as their difficulties in challenging the mathematical content of their teaching.

III. Discussion and conclusion: Learning mathematics through research

In this article we have reviewed reports of studies involving teachers as researchers, showing that, in the main, teachers’ research focus on pedagogical issues related to their teaching. We then reported a study in which teachers researched aspects of MfT limits in secondary schools in Mozambique. This research involved both mathematical and pedagogical aspects of limits of functions, with this key mathematical concept in the foreground. The study provides evidence that the teachers’ mathematical knowledge could not be taken for granted. It also highlights two main difficulties that teachers faced during this process.

Firstly, for those aspects of MfT that required a deep understanding of some crucial mathematical concepts – in this case the graphical representation and the ε-δ definition – the evolution of teachers’ mathematical knowledge through the research process was constrained. Regarding the graphical register, the only teacher whose knowledge evolved substantially was David, whose research topic was directly linked to working with graphs. As a consequence, he benefitted from more systematic learning about this issue during his research and from more explanations from his supervisor during discussions of his work in the supervision sessions. His colleagues’ work with graphs, in contrast, was limited to the discussions during the seminars. The issue of the ε-δ definition was not studied in a systematic way at all, by any of the teachers, and, as a result, we argue, all teachers continued to face difficulties with the definition and its meaning.

Furthermore, the research process was more challenging for the experienced Grade 12 teacher whose research also confronted the mathematical content of his own practice.

The study we report does not suggest that there is no place for teacher research in mathematics teacher education. Rather, it illuminates the centrality of the content of teaching, and so teachers’ mathematical knowledge for teaching. Teachers' participation in research, including the study reported here, enables the interrogation of practice. However, the content of the practice studied needs to be at the centre, as was the case with David and the graphical register. More specifically, we suggest that teachers get involved in research that increasingly puts mathematics at the core: research on MfT with attention to both mathematical and pedagogical issues and their intertwining in practice. This may help their knowledge regarding some aspects of MfT to evolve; at the same time, this would illuminate aspects of the content where they need more systematic teaching to overcome their difficulties. This systematic teaching should then be implemented.

In this way, teachers could produce knowledge that helps their personal relationship to mathematics and its teaching and learning to evolve, as well as improve their practice, and so realise the core expectation of the teachers-as-researchers movement. We do not claim that they would necessarily teach in a radically different way, as they would still be exposed to institutional constraints. Rather it is hoped their changed personal relationship to mathematics, that is to the actual content of their teaching, would enable them to teach and interpret the curriculum with more insight and a critical eye.

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Endnote

1 Pseudonyms are used for teachers' names.

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