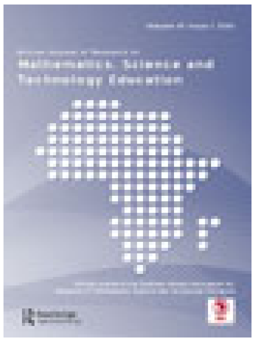

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A Framework for Describing Mathematics Discourse in Instruction and Interpreting Differences in Teaching

Jill Adler* and **Erlina Ronda**[1]

School of Education, University of the Witwatersrand, South Africa
**Corresponding author. School of Education, University of the Witwatersrand, South Africa. Email: jill.adler@wits.ac.za*

We describe and use an analytical framework to document mathematics discourse in instruction (MDI), and interpret differences in mathematics teaching. MDI is characterised by four interacting components in the teaching of a mathematics lesson: exemplification (occurring through a sequence of examples and related tasks), explanatory talk (talk that names and legitimates what comes to count as mathematics in a particular lesson), learner participation (interaction between teacher and learners and amongst learners) and the object of learning (the lesson goal). MDI is grounded empirically in mathematics teaching practices in South Africa, and theoretically in sociocultural theoretical resources. The MDI framework allows for nuanced descriptions of mathematics teaching and interpretations of differences in what is mathematically made available to learn.

Keywords: *Mathematics*; *classroom discourse*; *exemplification*; *explanatory talk*; *student participation*

Introduction

In this article we describe an analytical framework for studying *mathematical discourse in instruction*, MDI, and illustrate its use in interpreting differences in teaching. MDI is characterised by four interacting components in the teaching of a mathematics lesson: *exemplification, explanatory talk*, *learner participation* and the *object of learning*. Since mathematical objects such as numbers and functions are abstract entities, they need to be exemplified and explained. *Exemplification* and *explanatory talk* are two commonplace practices in mathematics teaching, occurring within patterns of interaction, and towards a particular goal. We refer to this goal as the *object of learning* (Marton & Pang, 2006).

In previous work, we conceptualized and used MDI to examine coherence across an example/task and its accompanying explanation (Venkat & Adler, 2012), and then across a sequence of examples/tasks and the accompanying explanatory talk within a lesson (Adler & Venkat, 2014). It was our empirical data across both primary and secondary classrooms that emphasized the need for such coherence, and our sociocultural orientation to learning that foregrounds the importance of teaching mathematics as a network of interrelated and well-organized concepts, or scientific concepts (Vygotsky, 1978). We now refine aspects of the framework for interpreting differences in practice. We describe and then use the full MDI analytical framework to analyse *two lessons of one teacher* —one observed and video-recorded at the start (February 2012) and the other after her completion of a mathematics professional development course (March 2013). We use this case to illustrate the MDI analytical framework and then argue why it provides for useful and nuanced interpretations of mathematics made available to learn and differences in this over time.

Locating MDI

In their review of 30 years of research on mathematics teachers, teaching and teacher education, Ponte and Chapman (2006: 488) conclude with a call for future research that attends to 'innovative research designs to deal with the complex relationships among various variables, situations and circumstances that define teachers'… activities'. The framework offered in this paper responds to this call. Our focus is a framework that illuminates the complexity of teaching mathematics in ways that are productive in professional development research *and* practice. We sought a framework that characterizes teaching across classroom contexts and practices, foregrounds the mathematics made available to learn, and enables interpretations of difference, and ultimately shifts, in this over time.

There are strong commonalities with other recently developed frameworks that foreground the observation of mathematics in classroom research, the most relevant of which is the Mathematical Quality Instruction (MQI) framework developed in the Learning Mathematics for Teaching Project (LMT, 2011). The MQI framework emphasises the importance of generality in mathematics, and what mathematically is made available to learn, and thus shares our concern with making visible the mathematical quality in a lesson. MQI offers a broad range of indicators for the mathematical quality of instruction, and a clear description of how codes for analysis were combined into measures of a range of elements of instruction, measures that were then validated. The theory or lens that produces these descriptions and measures appears to be a view of mathematics, one that is not explicitly elaborated but rather taken as shared; similarly so with the privileged pedagogy underlying the framework. MDI is theoretically framed, and its elements are a function of the interaction between our theoretical field (broadly Vygotskian or sociocultural) and our empirical field—secondary mathematics classrooms in relatively poor schools in contemporary South Africa. It is our theoretical framing that enables us to identify and justify the more contained but nevertheless key features of instruction that together (a) do justice to the mathematics that is made available to learn and (b) enable the interpretation of differences in the mathematics made available to learn, interpretations that are framed by our privileging of teaching towards scientific concepts and how this unfolds over time.

MDI includes a focus on tasks together with examples as *exemplification*, the importance of both naming and legitimation in *explanatory talk*, and *learner participation.* As discussed, these mediational means are commonplace in teaching, and a productive starting point for working on practice. Our intention in this paper is to substantiate the claim that MDI contributes to bringing the mathematics made available to learn clearly into view.

MDI Framework—Theoretical Resources and Analytical Tools

Object of Learning

Our starting point was that learning is always about something. Bringing into focus what this is, in terms of what learners are expected to know and be able to do, is central to the work of teaching. Marton and Tsui (2004, p. 228) refer to this 'something' as the *object of learning*, and posit: 'The object of learning … is defined in terms of the content itself … and in terms of the learner's way of handling the content'. The usefulness of foregrounding a connection between 'object' and 'learning' (instead of simply referring to the lesson 'goal') is that it focuses attention on both the content *and* what learners are expected to be able to do with respect to that content, that is, capabilities. An object of learning in a mathematics lesson could be a concept, procedure or algorithm, or meta-mathematical practice. It goes without saying that the object of learning needs to be in focus for the teacher. Bringing this into focus for learners, and from a Vygotskian perspective of enabling the building of scientific concepts, is not trivial. In our MDI framework captured in Figure 1, the key mediational means or generative mechanisms for this work of teaching are *exemplification*, *explanatory talk* and *learner participation*. What stands between (i.e. mediates) the object (and here *of learning*) and the subject (the learner) are a range of cultural tools such as examples and tasks, word use and the social interactions within which these are embedded.

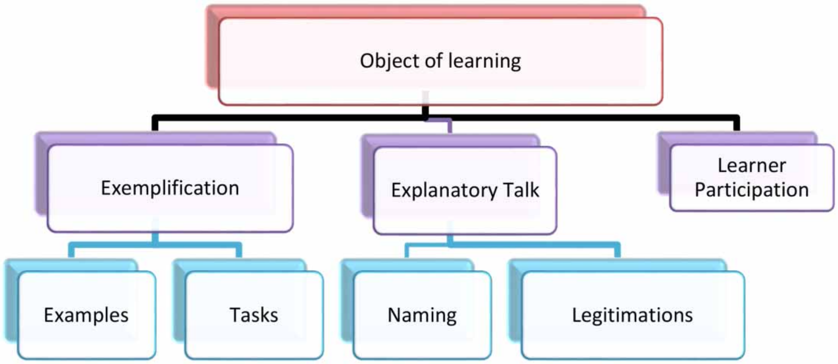


Figure 1: Constitutive elements of MDI and their interrelation

We now move on to define and operationalize the remaining inter-related and constitutive elements of MDI. We have combined our theoretical elaboration with related analytical tools, as these processes have been deeply intertwined in our work. As we worked between concepts and data, so our analytical tools were refined. What is not captured in Figure 1, as a static representation, is that teaching and learning occur in time and over time.[2] We attend to the temporal unfolding of a lesson and what is made available to learn through our unit of analysis and analytical framework.

The Analytical Process and Unit of Analysis

We want to describe the mathematics made available to learn during instruction—and so describe what is enacted. Our unit of study in the first instance is a lesson. Our analysis is on a detailed transcription of the video-record of the lesson. Analysis requires manageable 'chunks' that support interpretation of exemplification, explanatory talk and learner participation, in relation to the object of learning. Our first step, having developed some familiarity with the lesson, is to identify the intended object of learning. As is well known, what is intended is not synonymous with what is enacted (Marton & Tsui, 2004). We infer the intended object of learning by what is announced in some way, typically by the teacher, at the start of the lesson, often stated as a topic or written as a heading on the chalkboard (e.g. Dividing Fractions). We then chunk the lesson transcript into *mathematical episodes*, so named because they are identified by a shift in content focus, typically marked by a task that encompasses selected example(s), and bears some relation to the stated object of learning. A next episode is identified by the introduction of new task and so different 'content'. The dividing of the lesson into mathematics episodes as the *unit of analysis* enables us to consider example sets related to a task, and the accompanying explanatory talk, as well as how these accumulate over the lesson. In this way we produce a description of what is mathematically made available to learn (or the enacted object of learning) within and across episodes.

Exemplifying

Examples

Attention to *examples* has a long history within both mathematics and mathematics teaching. Bills et al. (2006) provide a substantial review of literature of examples from which they conclude that further research needs to attend to the sequencing of, and variation across, a succession of examples as well as teachers' attention to choosing and using examples in their teaching. We agree, more so because as suggested earlier and confirmed in empirical studies, examples are ubiquitous in mathematics teaching, and typically the focus of explanatory talk. We use Zodik and Zaslavsky's (2008, p. 165) definition of an example as 'a particular case of a larger class, from which one can generalize'. They too observed that examples were commonplace in secondary mathematics lessons. Interestingly, teachers in their study were not conscious of their selection strategies and example use.

Zodik and Zaslavsky called for research into the deliberate use of examples as a focus in teacher preparation.

The research on examples illuminates what teachers do and why, but does not enable a view of how examples accumulate to bring the object of learning into focus for learners, and whether there is movement towards generality. Careful analysis is needed. Two necessary aspects for a sequence of examples that constitute a basis for generalization are *similarity and contrast*. We draw on the work of Watson and Mason (2006), who attend to variation amidst invariance in mathematics, and on variation theorists (e.g. Marton & Tsui, 2004), to categorise three different patterns of variation: *similarity* (S), *contrast* (C) and *fusion* (F). On the basis of our empirical data, we consider how these operate separately and together over the full sequence of examples in a lesson. Focusing on what something is through a set of *similar examples* brings possibilities for generalizing that which is invariant. Similarity on its own, however, does not draw attention to the boundaries around a concept, and so to what it is not. *Contrasting examples* that bring attention to a different class also make available opportunity for generality. Moreover, further generality is possible through *fusion*—when more than one aspect of an object of learning is simultaneously varying/invariant across an example set. We categorize the examples within a mathematical episode. We then look across episodes at how similarity, contrast and fusion are at work, and through this describe movement towards generality across a lesson.

For example, in the lesson analysed later in this article, the four examples below were presented in one episode—simplifying algebraic (rational) expressions.

Example 1: $\frac{1}{2}$

Example 2: $\frac{3+7}{3}$

Example 3: $\frac{3a^2+6a}{6a}$

Example 4: $\frac{(x-2)(x+1)+3(x-2)}{x-2}$

Example 5: $\frac{2x^2+x-6}{2x^2+4x}$

All five examples are rational expressions. Examples 2–4 have similar *structures*, a binomial divided by a monomial, *with terms varying* (numerical, algebraic, simple, complex, i.e. with common factors); and with *contrast* in that there are common factors in the numerator and/or denominator in Examples 3–5, but not in Example 2, nor in the simplified term resulting in Examples 3 and 5. Moreover, in Example 5 in particular, factoring, identifying common factors, and recognising the resulting simplified term as not having common factors occur *simultaneously*. Hence we code this episode as exhibiting S, C and F.

We then look across episodes to produce a *summative judgment*, expressed as a *level* with respect to the example set that accumulates over the lesson. Judgment is made in relation to movement towards generality. We judge an example set as at Level 1, for example, if the sequence of examples displays one form of variation, that is, similarity or contrast (S or C) and so the opportunity for generalizing in relation to one aspect of the object of learning; OR the opportunity for seeing at least one instance of what is not the object. We judge it as Level 2 if at least two forms of variation are displayed and so the opportunity for generalizing two aspects of the object of learning (S and S OR S and C); and we judge it as Level 3 when there is fusion—simultaneous variation of more than one aspect of the object of learning *and this builds on or is connected with similarity and contrast within the example* *set (S, C, F)*. These three levels thus do not apply to an example set where, from the outset, multiple aspects of a concept are varying at the same time, and there is no specific attention to these aspects. Such example sets are coded Level 0, indicating there is no explicit attention to building generality.

The codes and levels of all the elements of the MDI framework are summarized in Table 1 with those on examples in the first column. We emphasise here that our conceptualizing of levels is to enable interpretation of differences—hence our reference to summative judgments—and of progress terms of what is mathematically made available to learn. The levels here do *not* suggest a teaching sequence.

Tasks

Examples do not speak for themselves. There is always a task associated with an example, shaping what it is learners are to do. We define task as simply what learners are asked to do with the various examples presented. Thus while examples are selected as 'particular instances' of the general case in focus, and for drawing attention to 'relevant features', tasks are designed to bring particular capabilities to the fore (Marton & Pang, 2006; Marton & Tsui, 2004). For example $2x - 7 = -x + 5$ is an example of a linear equation. Associated tasks with respect to this example can vary from 'solve for x' to 'construct a different equation in x where $x = 4$ is the solution'. These tasks require different actions, at different levels of complexity or cognitive demand, and so make available different opportunities for mathematics learning.

Notwithstanding the abundance of recent literature and research on task design in mathematics education (e.g. Shimizu, Kaur, Huang, & Clarke, 2010; Watson & Ohtani, 2015), we confine ourselves to the distinction between low and high cognitively demanding tasks, where increasing cognitive demand is seen in terms of the extent of connections between and among concepts and procedures. We distinguish tasks that require learners to carry out *a known operation or procedure* (K), from those that require *application of what is known in relation to the object of learning* (A). In a task requiring application, *decisions are needed for steps to carry out*. In addition to K and A tasks, we distinguish tasks where the demand is for *multiple connections and problem solving* (C/PS). These three categories correspond to the three levels described in Table 1. As with Stein, Smith, Henningsen and Silver (2000), we found movement between the set-up of a task, and how it was carried out by teachers and learners. We have many tasks across lessons that invite A or MC/PS, but that are ultimately demonstrated or carried out by the teacher. We categorized these, for example, as A → K, tasks that ultimately become low-level tasks for the learners. If this predominated across episodes, we judged the accumulated task level as Level 2 → Level 1.

Explanatory Talk

Our emphasis on *explanatory* talk draws on earlier work in the Quantum project (Adler & Davis, 2006; Davis, Adler & Parker, 2007) and Bernstein's (2000, p. 36) insight that 'key to pedagogic practice is continuous evaluation'. For Bernstein, any pedagogic discourse, hence the discourse in a mathematics lesson, transmits criteria as to what counts as mathematics. The transmission of criteria occurs continuously, be it implicitly or explicitly, through messages that are communicated as to what is valued with respect to the object of learning, that is, what is to be known or done, and how. We call this *explanatory talk*, the function of which is to name and legitimate what is focused on and talked about, that is, related examples and tasks. Analysing how objects[3] focused on are named, and what is legitimated in an episode is key to being able to describe the mathematics made available to learn through explanatory talk, as well as reach a summative judgment on naming and legitimating as these accumulate over time in a lesson.

Legitimating criteria

The Quantum research identified different domains of legitimation: the domain of mathematics, non-mathematical domains, the curriculum and the authority of the teacher. Work in school classrooms led to elaboration of the mathematical domain, and distinctions between criteria related to *properties of mathematical objects*, *accepted conventions* and *derived procedures*, *instances or empirical*

Table 1: Analytical framework for MDI

Object of learning				
Exemplification		Explanatory talk		
Examples	Tasks	Naming	Legitimating criteria	Learner participation
Examples provide opportunities within an episode or across episodes in a lesson for learners to experience variation in terms of *similarity* (**S**), contrast (**C**), fusion (**F**)	Across the lesson, learners are required to: *carry out known operations and procedures* (**K**) e.g. solve for *x*, multiply, factorise; *apply known skills, and/or decide on operation and/or procedure to use* (**A**), e.g. compare/classify/ match representations; *use multiple concepts and make multiple connections* (**C/PS**), e.g. solve problems in different ways, use multiple representations, pose problems, prove, reason, etc.	Within and across episodes word use is: *colloquial* (**NM**), e.g. everyday language and/or ambiguous pronouns such as this, that, thing, to refer to objects in focus; *math words used as name only* (**Ms**), e.g. to read string of symbols; *mathematical language used appropriately* (**Ma**) to refer to other words, symbols, images, procedures, etc.	Legitimating criteria: *non-mathematical* (**NM**) *visual* (**V**), e.g. cues are iconic or mnemonic; *positional* (**P**), e.g. a statement or assertion, typically by the teacher, as if 'fact'; *everyday* (**E**). *Mathematical criteria:* *local* (**L**), e.g. a specific or single case (real-life or math), established shortcut, or convention; *general* (**G**) equivalent representation, definition, previously established generalization, principles, structures, properties, which can be partial (**GP**) or 'full' (**GF**)	Learners answer *yes/no questions or offer single words* to the teacher's unfinished sentence **Y/N**. Learners answer (what/how) questions in phrases/sentences (**P/S**). Learners answer why questions; present ideas in discussion; teacher revoices/ confirms/asks questions (**D**)

The set of examples provide opportunities in the lesson for learners to experience: Level 1—one form of variation, i.e. S or C; Level 2—at least two forms of variation, S and S OR S and C; Level 3—simultaneous variation (fusion) of more than one aspect of the object of learning and connected with similarity and contrast within the example set (S, C, F). Level 0: simultaneous variation with no attention to similarity and/or contrast	Tasks provide opportunities for: Level 1—carrying out known procedures only (K); Level 2—K and/or some application A; Level 3—K and/or A and C/PS L2 → L1: A → K or C/PS → K is assigned to tasks set up at level 2 or 3 but then reduced to 1 when it unfolds	Use of colloquial and mathematical words: Level 1—NM, there is no focused math talk, all colloquial/everyday; Level 2—movement between NM and Ms, some Ma; Level 3—movement between colloquial NM and formal math talk Ma	Criteria for what counts as mathematics that emerge over time in a lesson and provide opportunity for learning geared towards scientific concepts. Level 0—all criteria are NM, i.e. V, P, E; Level 1—criteria include L, e.g. single case; Level 2—criteria extend beyond NM and L to include generality, but this is partial GP; Level 3—GF math legitimation of a concept or procedure is principled and/or derived/ proved	Opportunity for learners to speak and so use math discourse is at: Level 1—Y/N only (single words only); Level 2—at least some P/S in more than one episode (phrases and sentences); Level 3—P/S and at least some D (discussion) in more than one episode

cases, and then the *general case or proof* (Adler, 2012). While these categories emerged from analysis in particular empirical fields, they provided initial analytical tools for categorizing different kinds of criteria in mathematics lessons more generally, and we proceeded in MDI as follows. We distinguish criteria of what counts (or not) as mathematical that are particular or localized (L) (e.g. a specific or single case, an established shortcut, or a convention) from those that have some generality (e.g. equivalent representation, definition, previously established generalization; principles, structures, properties), distinguishing partial (PG) from full generality (FG). We are also interested in non-mathematical criteria (NM), everyday knowledge or experience (E), visual cues (V) as to how a step, answer or process 'looks' (e.g. a 'smile' as indicating a parabola graph with a minimum; the distributive law illustrated with an arrow connecting the each term 'inside the bracket' to the number 'outside', or memory devices that aid recall (e.g. FOIL)); or when what counts is simply stated, thus assigning authority to the position (P) of the speaker of the statement, typically the teacher.

The significance of these varying criteria is the opportunities they open and close for learning. Most obvious are the extremes of legitimations based on the one hand on principles of mathematics, thus with varying degrees of generality, and possibilities for learners to reproduce or reformulate what they have learned in similar and different settings. On the other hand, appeals to the authority of the teacher and so legitimations based on position (rather than principle) and/or visual cues produce a dependency on the teacher, on memory (this is what you must do), or on how things 'look', requiring imitation that is local or situational (Sfard, 2008). While imitation might be necessary in aspects of mathematics learning (Vygotsky, 1978; Sfard, 2008), these imitations cannot be the endpoint of learning. The criteria for what counts as mathematics that emerge over time in a lesson are thus key to what is made available to learn in terms of movement towards scientific concepts.

Naming

Learners' encounters with mathematical objects also occur through how these are named. Naming focuses learners' attention in particular ways. A critical element of talk in the mathematics classroom is thus how objects focused on are named. We define naming to mean the use of words to refer to other words, symbols, images, procedures or relationships. In her study of teachers' knowledge of their practices in multilingual mathematics classrooms Adler (2001) identified an ever-present tension between the importance of formal mathematical talk if learners were to be inducted into mathematical discourse, and its alienation if disconnected from learner thinking. The tension in managing both formal and informal ways of talking mathematically, and thus naming objects focused on in class, is now widely recognized. Sfard (2008), for example, distinguished formal mathematical discourse from colloquial discourse, arguing that, while colloquial discourse is a necessary part of the learning process, it is participation in formal discourse that ultimately marks out learning mathematics. In our project work, we noticed some teachers' reluctance to use formal mathematical language as it is 'abstract and the learners are put off', and others' over-reliance on formal talk with neglect of connecting to colloquial meanings. Hence our attention to naming in MDI.

We categorise naming within episodes as either *colloquial* (NM, non-mathematical, and here we include everyday language, e.g. 'over' in division, and/or ambiguous pronouns such as this, that, thing, to refer to objects acted on) or *mathematical* (M), but in the latter distinguish mathematical words used as labels or to read a string of symbols (Ms) from formal mathematical language used (Ma). For example, in episode 4 in Table 3, 'over' is categorised as non-mathematical, despite its common use in school mathematics instruction. This is not because the word 'over' should not be used when working with division. Our point is simply that, if the word 'over' is used exclusively to describe a fraction, then a fraction as a division is never made explicit. The focus is on appearances, not relations. While appearance is important for writing a fraction, it does not convey the relation of division that defines a fraction. Our purpose is to see the extent of both colloquial and formal talk and the movement between these. Our summative judgment as a 'higher' level of naming depends on movement across colloquial and formal mathematical naming in the lesson.

Table 1 summarises the categories and coding for explanatory talk and the levels assigned to these across a lesson. The levels reflect our privileging of mathematical names and principled criteria.

However, and this is critical in these components of MDI, these are ultimate goals and realized through fluid movement between categories. These should skew towards higher levels when mathematical work related to the object of learning moves to consolidation, for example, in the later part of a lesson or lesson sequence, or when there is a review of a previous lesson.

Learner Participation

In addition to the mediational means discussed so far (examples, tasks, naming, legitimating criteria), we are concerned with what learners are invited to say, and specifically whether and how learners have opportunity to speak mathematically and to verbally display mathematical reasoning. Learner participation in the framework allows us to describe what learners are invited to say apart from the tasks. We view this as important in multilingual contexts. As we only used one camera, we did not capture learner-to-learner discussions or individual work. What is visible in the video lesson is how the teacher engaged with learners during whole class discussion and in some instances where the teacher talked with individual learners as she/he inspected their work. Codes and levels here are provided in the last column in Table 1.

The Framework

Table 1 summarizes the MDI framework, its components and how we prise these apart through coding categories. Our summative judgments in levels show that we privilege increasing generality across examples, increasing task demands, movement between colloquial and formal mathematical naming, and criteria that are grounded on mathematical principles, properties, etc. The levels imply a hierarchy with respect to generality and formal mathematical discourse. We repeat here that these are for describing differences in teaching and progress in what is made available to learn. They *do not suggest a teaching sequence*.

We now use our framework to describe two lessons, a year apart, and MDI over time. Before we do this some caveats are needed. Firstly, these are two single lessons, separated over time. These do not and cannot reflect Ms X's teaching in general. How this particular teacher works with exemplification and explanatory talk *within and then across the lessons* is illustrative of our interpretation of differences in what is made available to learn. Secondly, space constraints lead us to describe and analyse the 2013 lesson in some detail, and contrast it with a summary analysis of the 2012 lesson.

MDI in Ms X's 2013 Grade 10 Lesson

Ms X began the lesson by stating it was on 'Revision of algebraic fractions'. There were five mathematical episodes corresponding to a changing content focus: revising terms and factors (Episodes 1 and 2), simplifying algebraic fractions (Episode 3) leading to a focus on the operation of division of algebraic fractions (Episode 4). Episode 5 involved the equivalent expressions for fractions with negative coefficients. We thus interpreted the overall intended object of learning as *Dividing Algebraic Fractions (i.e. rational expressions).* Exemplification in each of Episodes 1–5 is summarized in Table 2.

Exemplification

Within and across Episodes 1, 3, 4 and 5, with respect to *exemplification*, there was consistency in approach. For instance, in all *example sets*, these progressed from numerical to algebraic with increasing complexity of expressions in terms of the number of terms, the number of operations and the factorization involved. The gradual introduction of features that varied attracted attention to what was varied and thus opportunities for learners to see what remained invariant. In each episode there was attention to observing generality through similarity (S). Contrast (C) was also used to show what is not a single term in Episode 1, and when cancelling a common factor and term is permitted (Episode 3). The last expression in each of Episodes 1, 3, 4 and 5 presented more than one aspect varying together and so the opportunity for learners to experience fusion (F). Our summative judgment of the set of examples across the lesson, and with respect to the object of learning being the division of algebraic fractions overall was Level 3, that is, simultaneous variation of more than one aspect of the

Table 2: Examples and Tasks—Ms X's 2013 lesson

Episode and codes	Examples, tasks and comment
Episode 1 Revision of meaning of terms **Examples:** S, C, U **Task:** K	Example 1: $x + x + 4x + y$ Example 2: $x + y$ Example 4: $2(x + y)$ Example 5: $\frac{2(x+y)}{2}$ Example 6: $\left[\frac{2(x+y)}{2}\right]^2$ Example 7: $\left[\frac{2(x+y)}{2}\right]^2 \times ab$ *How many terms?* Ms X discussed example 1 and its four terms, defining terms as separated by a 'plus or minus sign'. She then presented Examples 2—7 one at a time and asked the class as a whole to say how many terms there were in each expression. The emphasis for Examples 4–7 was on the operation carried out (multiplication, division, raising to a power) through which the expression remained a single term (as these were not plus or minus)
Episode 2 Revision of common factor **Examples:** NA **Task:** K	Example 1: $ax + bx$ *What is a factor? What is the common factor (Example 1)?* Ms X had learners respond to these questions. She revoiced contributions that a 'factor is a number that divides another without remainder', and common as being 'in both terms'
Episode 3 Simplifying algebraic fractions **Examples:** S, C, U **Task:** A $\rightarrow$ K	Example 1: $\frac{1}{2}$ Example 2: $\frac{3+7}{3}$ Example 3: $\frac{3a^2+6a}{6a}$ Example 4: $\frac{(x-2)(x+1)+3(x-2)}{x-2}$ Example 5: $\frac{2x^2+x-6}{2x^2+4x}$ *Can I say 3 and 3 here is a common factor (Example 2)?* After explaining that 3 and 7 are 'two terms', and 3 was not a common factor, Ms X went through each of examples 3–5, one at a time, asking the whole class specific questions like 'what is common factor here?' (in the numerator of Example 3), leading the class through steps to the solution (factoring numerator/denominator, then cancelling common factors)

Episode 4 Dividing algebraic fractions **Examples:** S, U **Task:** A →K	Example 1: $\frac{2}{6} \div \frac{2}{3}$ Example 2: $\frac{2x}{6x} \div \frac{2x}{3x}$ Example 3: $\frac{x^3 - x^2}{4} \div \frac{x^2}{8}$ Example 4: $\frac{x^2 - x}{x^2 + x - 2} \div \frac{x^2 + 4x}{x^2 - 4} \times \frac{3x + 12}{1}$ After revising the rule and procedure ('swap over' and multiply, then cancel common factors) for the division of two numerical fractions, Ms X proceeded with Examples 2 and 3, emphasizing the structure of the division of one fraction by another by keeping this form constant while varying terms. Examples ranged from simple to complex; numerical to algebraic. Example 4 extended to three fractions and a product. Examples 3 and 4 associated common factors with fraction division (and so simultaneous (U) attention to two features)
Episode 5 Interpreting expressions negative coefficient **Examples:** S,U **Task:** A →K	Example 1: $-\frac{4}{2}, \frac{-4}{2}, \frac{4}{-2}$ Example 2: $-\frac{a}{b}, \frac{a}{-b}$ Example 3: Is $-\frac{a-c}{b}$ Example 4: $\frac{a-4}{2(4-a)}$ *Negative 4 over divided by 2, how much is this going to give me?* Ms X went through each set of fractions in Examples 1 and 2, changing the position of the negative sign explaining that these had the 'same effect'. She then introduced additional terms in the numerator (Example 3) and went through similar steps for Example 3, and steps to simplify the common factor in Example 4

object of learning *and* this built on and connected with similarity and difference within the entire example set.

Each set of examples involved different *tasks*. In Episode 1, the task was to determine the number of terms for each example. Episode 2 was simply a reminder about a common factor and then factorising a binomial expression. In Episode 3, the task was to simplify expressions by first factorising expressions and then cancelling common factors. Episode 4 was focused on a rule and procedure for dividing algebraic fractions. These tasks provided learners with different ways of engaging with rational expressions. From the MDI analytical tool, we could judge the tasks across the lesson as Level 2 (opportunities for application). However, the teacher did *all* of the tasks herself, limiting the task for learners to respond to questions on individual steps, each of which then involved carrying out known procedures (K) related to the object of learning. Each episode was categorized as A→K, hence our summative judgment as Level 2 →1.

Explanatory talk

We focus our analysis of naming and legitimating criteria on Episode 4 as this was the episode on division of algebraic fractions. There was similarity in approach across the episode. In Table 3, column 1, we present the codes for naming and criteria assigned for Example 3, with the transcript of the talk in the middle column, and boardwork in column 3. This extract enables us to illustrate our coding, hence its choice. As we look at Example 3, we need to recall comments on Examples 1 and 2 in Table 2, where Ms X built a definition of a term through successive examples, and revised simplifying fractions, numerical and algebraic, where there were common factors, also defined. The procedure for division was expressed as the rule 'swop over and multiply'.

The *naming* codes in Table 3 show that, while there was frequent non-mathematical talk through the use of ambiguous pronouns (e.g. this, thing Lines 1, 37), talk was also mathematical, mostly reading of strings of symbols (e.g. Lines 1, 20) and occasional appropriate formal naming of objects (e.g. Line 5). This typifies the naming across all the examples in Episode 4, and our summative judgment of this episode as NM and Ms, as summarized in Table 4.

The overarching *legitimating criteria* in this part episode were to previous examples as the 'same thing' and their general structure—one algebraic fraction divided by another (Lines 1, 5, 14, 37; GF). The 'top' and 'below' (V) of the fractions were pointed to as needing to be 'one term' (Ma), and so expressed as factors that were defined in Episode 2 as 'dividing without remainder' (GF). The division follows a short cut (L; Lines 30—36; change signs and 'swap') with rules and procedures (factorise first, take out common factor, I cannot just go and say …; Lines 8,11, 37) that were stated, not derived (P). In overview, the criteria for recognizing the *form of the expression* were fully generalized, but the criteria for the procedure for division were dominantly localised, as there was reliance on rules, shortcuts, and in some cases Ms X's authority.

With respect to *Learner participation*, there were occasional learner answers to what/how questions in phrases or sentences. The dominant form was yes/no answers or supplying words to Ms X's unfinished sentences. The rare occasions where Ms X asked a why question that could have invited learners into greater participation in the discourse, she answered the question herself (Line 16).

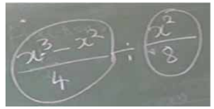
Our analysis of the MDI in Episode 4 shows an example set with S and F, but tasks reduced to K (carrying out known procedures), and participation Y/N. Naming moves between colloquial (NM) and mathematical, both Ms and Ma. Criteria shift between emphasis on visual features of the expression (NM), established conventions/short cuts (L), with some structure and generality (GP).

Table 4 provides the coding of the elements of MDI within and across the five episodes in Ms X's 2013 lesson. The final row contains our summative judgment, as a level, of what was mathematically made available to learn.

Across the 2013 lesson, Ms X made use of the three patterns of variation to show the generality of the object of learning, in this case, the structure of the division of 'algebraic fractions'. Her set of examples linked numerical fractions and there was gradual progression in the complexity of the terms in the algebraic fractions. Various ways of engaging with algebraic fractions were demonstrated: making available the recognition of an algebraic fraction as a single term (Episode 1); that algebraic fractions

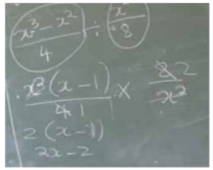
Table 3: Analysis and coding of explanatory talk, Episode 4, Example 3:

$$\frac{x^3 - x^2}{4} \div \frac{x^2}{8}$$

Line, code	Transcription[a]	Boardwork
1–5 **Criteria:** GF, structure **Naming:** NM, Ms, Ma,	T: It's ***one and the same thing***. They give you *something like this* (writes symbols on board), … x cubed minus squared *the whole thing over, over* four divided by *x* squared *over* eight … OK? LS: Yes T: So it's, it's *one and the same concept*. Over here (points to $\frac{2}{6} \div \frac{2}{3}$) you just have two numbers, a fraction divided by a fraction, OK? LS: Yes T: Over here (pointing back to Example 3) is ***the same thing.*** I've got, here's **one fraction divided by one fraction** (**circles each fraction**) …	
6–20 **Criteria:** GF, rule NM, V **Naming:** NM, Ma	T: So … . What are we going to do *over here*? (points to first fraction in Example 3). LS: (some) we are going to divide. T: … **remember the rule that we learnt over there**? (points to factoring, **defined in Episode 2**). LS: Yes. T: Before we can go and divide, what must I do? LS: *Take out* the common factor T: *Take out* the common factor, OK? LS: Yes T: So, **the *same thing applies here*.** It is everything that you, that you have learned, *but they just put it into one thing* to make it look a bit complicated. It's actually very simple, OK? LS: Yes T: **So, *over here* it concerns the common factor. Why? Because *we want to have one, one term at the top and one term below***, OK? LS: Yes T: So, what is the common factor to the two terms? Ls: *x* squared	

(*Continued*)

Table 3: Continued

Line, code	Transcription[a]	Boardwork
21–36	T: *x* squared, *open the bracket*, but … *x* squared *from x* cubed	
Criteria: L, Short cut	T (with Ls inserting words to complete what T said … (one word turns by T and then Ls not included here)) *x* minus … 1 … *the whole thing* divided by? … four	
Naming: NM, Ms	T: Now? Ls: Times **… change the sign** T: **The same thing, change the sign**, OK? … and then 8 over *x* squared	
37–39 **Criteria:** GF, structure NM, P NM, V **Naming:** NM, Ms	T: So, you **just apply the same principle**, it's just that when it looks complicated just pause and say what must I do here? **Because I know terms *like this* (points to $\frac{x^3 - x^2}{4}$), I cannot just … go and say *this* (pointing to $x^3 - x^2$) divided by *this* (points to 4) …** OK**?** LS: Yes T: **Make sure that *you have got* one term *at the top and* one term *below*.** So from here I can, what must I do … (pointing to the first fraction)? Ls … inaudible … (is common) … *x* squared … (This part of Episode 4 ends with a similar interactional pattern and use of words by Ms X and learners, carrying out next steps (factoring, cancelling common factors, simplifying) to produce the answer)	

Key: T, Ms X; L, learner; Ls, learners in unison; *italics for colloquial*, underlining for formal language and **bold type for legitimating criteria**, categorized as discussed above.

Table 4: Coding elements and summative judgment of MDI (Ms X's 2013)

Episodes	Examples	Tasks	Naming	Legitimating criteria	Learner participation
1, Meaning of a term	S, C, F	K	NM, Ms, Ma	GF	Y/N
2, Meaning of common factor	NA	K	Ms, Ma	GF	Y/N, P/S
3, Simplify algebraic fraction	S, C, F	A → K	NM, Ms	NM, L	Y/N
4, Divide algebraic fractions (+)	S, F	A → K	NM, Ms, Ma	NM, L, GP	Y/N
5, Extension to (–) coefficients	S, F	A → K	Nm, Ms	L	Y/N
Summative judgment	L3	L2 → L1	L2	L2	L1

may not always be in simplified form and hence may be simplified just like numerical fractions (Episode 3); that like numerical fractions, operations may be performed on algebraic fractions (Episode 4); and that like their numerical counterpart, a negative sign in the numerator or denominator or alongside the fraction itself represented the same algebraic fraction. The legitimating criteria where she repeated, 'it's one and the same thing', 'remember the rule that we learnt over there', 'it's one and the same principle' all referred to the previously established forms and procedures with the numerical fractions, make available generality in the structure division of algebraic fractions, and in procedures, albeit that some were short cuts.

At the same time, Ms X demonstrated the procedure in all examples, reducing the learners' task to arithmetic operations and participation to answering yes/no or completing sentences with single words or short expressions. Ms X's word use moved between colloquial and formal mathematical talk within and across episodes, with a predominance of reading out strings of symbols. Here too learners were not provided with the opportunity to talk in elaborated ways. Thus, while opportunity to learn division of algebraic fractions was opened through the example set, this was not supported by the tasks and only partially by the explanatory talk, reducing possibilities for learners' successfully dividing algebraic fractions independently.

MDI in Ms X's 2012 Grade 10 Lesson

A year earlier, February 2012, Ms X's lesson was on 'Applying Laws of Exponents', and made up of two episodes. Episode 1 revised the laws of exponents: each of five laws was captured on the chalkboard by Ms X, with learners contributing in unison the various components of the laws, completing her sentences. Episode 2 spanned most of the lesson and involved applying the laws of exponents to simplify exponential expressions with different numerical and then literal bases and exponents, as well as numerical coefficients. Most examples also required the use of more than one Law (division and multiplication for example). The examples across Episode 2 became more complex in terms of simplification steps, and so increased task demand, all involving recognition and application of revised laws. Not all laws were revised at the outset and so some were inserted; multiple features of an exponential expression were varied simultaneously, following the first example involving the multiplication law.

The presentation of the examples in Episode 2 can be divided into two parts, the first focused on the numerical and the second on literal bases and exponents. In Episode 2.1 Ms X wrote the example on the board, and asked learners to 'simplify' using the Laws discussed. Learners were given time to work on these tasks on their own or with a partner, but soon Ms X saw that most were not making progress. She called the class together and went through examples of simpler tasks (e.g. looking only at base 2 components of the expression in Episode 2.1, then base 3), and so broke down the task into smaller parts. Ultimately, she demonstrated the prodedure for doing the original task, reducing each task from A (application) to carrying out known procedures (K). Learners had the opportunity to talk about the tasks initially themselves. Ms X's word use when talking about these was predominantly formalised, reading out strings of symbols, with some generality in legitimating criteria. Space constraints preclude detailed transcript analysis. The lesson episodes are summarised in Table 5, and coding in Table 6.

Table 5: Lesson 2012, Examples, Tasks and comments across episodes

Episode	Examples, Tasks, comment
1, Revision of exponent laws **Examples:** NA **Tasks:** K	$a^n \times a^m = a^{n+m}$; $a^m/a^n = {}^{m-n}$; $(a/b)^n = a^n/b^n$; $(ab)^x = a^x b^x$; $(a^m)^n = a^{mn}$ Each law was stated in words and captured on the board—accumulate into the set of laws expressed symbolically, and assumed already known. Note no recall here of laws for 0 or negative exponents, viz: $a^0 = 1$; and $a^{-n} = 1/a^n$
2, Simplifying expressions 2.1, with numerical bases and exponents	Example 1: Simplify: $3^0 \times 3^4$ Solution 1: $1 \times 81 = 81$ (Recall: $3^0 = 1$) Solution 2: $3^0 \times 3^4 = 3^4 = 81$ Example 2: Simplify $\frac{3 \times 2^2 \times 3^5}{3^{-3} \times 2^2 \times 3^9}$ (Recall: $a^{-n} = 1/a^n$)
2.2,with literal bases, numerical exponents, numerical factors	Example 3: Simplify $\frac{3ab^2c^3}{4ab^2c} \div \frac{4^{-3}c^3}{4c^7}$ (a) $\frac{1}{2} \div \frac{3}{2} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{2} \times \frac{2}{3}$ (b) $\frac{ax}{x^{-1}} \times \frac{x^2}{2a} \times \frac{2ax^4}{a^2}$ Example 4: Simplify $\frac{21p^{10}q^6}{-7qp^6} \div \frac{14p^4q^3}{r}$
Examples: F, S **Tasks:** A → K	Example 1 involves the multiplication law. Same numerical base of 3 and varying numerical exponents, including 0. Two solutions processes, result invariant. Example 2 includes multiplication and division, two different numerical bases, and negative and positive numerical exponents. Examples 3 and 4 involve three different literal bases, positive and negative numerical indices, and division of algebraic fractions with exponents; and Example 2 includes numerical factors. Examples 3a and 3b each inserted focus on one of the variations above: division by a fraction, negative numeric exponent with literal base

Looking from Table 6 (2012) to Table 4 (2013) and so over time and across the two lessons, we are able to see differences in exemplification, particularly with respect to the selection and sequencing of examples chosen. Ms X's set of examples in Episode 4 in 2013 evidences awareness of, and skill in, producing a sequence of examples that bring the operation of division with varying algebraic fractions into focus, hence the value of this specific aspect of MDI. Generality (of structure) is made available to learn through exemplification in 2013. This is in contrast with the 2012 lesson where examples jumped from relatively simple to complex, with multiple aspects varying simultaneously. That learners needed assistance was evidenced by the insertion of additional examples (3a, 3b). Overall, possibilities for generality were constrained.

There is also more movement between colloquial and formal talk in 2013. In 2012, there was little use of colloquial words and their revoicing in whole class discussion. Interestingly, there was greater learner participation particularly in Episode 2.1 in the 2012 lesson where learners were initially working on tasks on their own. Owing to the video record, we were not able to hear this talk. Tasks and legitimating criteria in the 2013 lesson remain similar to those in 2012.

Table 6: Coding elements and summative judgment of MDI (Ms X's 2012)

Episodes	Examples	Tasks	Naming	Legitimating criteria	Learner participation
1, Review exponent laws	NA	K	Ms, Ma	NA	P/S
2, Simplifying expressions	S, F	A, K	Ms, Ma	L, GP	P/S, Y/N
Summative judgment	L1	L2→ L1	$L2^-$	L2	L2

Discussion

In broad terms, one could argue that the two lessons are more similar than different, given Ms X's demonstration of most examples and tasks and the dominant explanatory talk. This would fail to recognize important differences in practice, differences that open up possibilities for a more elaborated enacted object of learning. The attention to variation amidst invariance, and so generality of structure in Ms X's lesson in 2013 is, in our view, significant, as is her revoicing and greater movement between colloquial and formal naming of objects. Our point here is that, by providing analytical distinctions within and between exemplification and explanatory talk, we are able to describe a range of mediational means and how they are at work separately and together. This is important for research and descriptions of practice, and more so for figuring out focused development work with teachers. For while her attention to examples and naming was different, at the same time, and critically so, much remained the same. Task demand remained low, and explanatory talk, particularly the criteria transmitted, were, in the main, either local or partial, and learner participation in mathematical discourse restricted.

We have used the framework here and illustrated it in detail on one teacher, as our purpose was to describe and illustrate the analytical frame. In published articles, when we report findings, space constraints result in brief descriptions of methodologies and analysis. We have thus focused this paper on the framework itself, its rationale and motivation. We will follow this with further reporting of results across teachers where patterns of difference might lead to claims about shifts in practice, and where we will need to address the challenge of comparability of different lessons, given our unit of analysis. Our goal here was to argue for the salience of the MDI framework.

There are limitations, as with any framework. Learner participation and tasks provide two views into learner activity. It might be possible to combine these and develop different categories that reflect both. We have stayed with 'naming' despite its more restrictive pointing to word use. This too could be developed further. These point to work for the future.

Conclusion

In this paper we have communicated the overall MDI framework, and illustrated its potential through analysis of selected project data. We recognize that MDI arises in a particular context. What then of its wider potential? Analytical resources are necessarily selective, reflecting a privileged reading of mathematics pedagogy. We have made these visible and explicit, and hold that MDI's generativity lies in its theoretical framing. We thus offer MDI—deliberately so named—as a further contribution to the problem of adequate and mathematically focused description of practice, one grounded in a different reality from much of the research in our field, one which we hope will contribute to the developing research on professional development, and ultimately how, if and when we can make claims about its worth.

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Notes

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2. Thanks to Lynn Slonimsky for the notion of learning happening ‘in time and over time’.
3. Our use of ‘object’ here is in the most general sense and includes all that is focused on, for example, words, symbols, images, pictures, material objects, etc.

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