CHAPTER 8: JILL ADLER AND ERLINA RONDA

A LESSON TO LEARN FROM

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Abstract

In this chapter we use a different lesson, a lesson that was taught by another teacher, Ms H, who also participated in the WMCS professional development (PD) project. It was taught more than a year after the video of Mr T's lesson was taken. In the intervening time, the WMCS teachers learned about the benefits of well-structured example sets for their students' learning in the PD. The teachers also had opportunity to work on what was a good explanation or justification for a particular concept or procedure, and on why the words you use to talk about mathematical ideas in class are important. Our purpose here is to move from the analysis in Chapter 5 to our interpretation of this for our PD work.

Keywords: Lesson study, professional development, math teaching framework, examples, naming, explanations.

From research insights to teaching

In Chapter 5, we analysed the opportunities made available to learn in Mr T's lesson. We focused on three key elements of teaching. We first examined the *examples* and related representations for solving the quadratic inequality in the lesson. We then looked at how Mr T and his learners used language to *name* the mathematical ideas they were talking about, and to *explain or justify* their solutions. Our analysis showed that while the selected examples and representations were similar to those offered in the curriculum documents and some textbooks, there were limitations in the accompanying explanatory talk. For example, the meaning of the solution to a quadratic inequality was not used as the basis for explaining steps taken in the process to solve the inequality. Also, the naming of symbols and procedures was often ambiguous or in everyday language. These ways of talking were not sufficient support for learners to solve quadratic inequalities independently. Learners would have to rely on their memory of what the teacher said, and on the visual appearance of various symbols and expressions – on how they looked, rather than on their meaning.

This chapter has a different purpose from that in Chapter 5. We are going to use a different lesson now, a lesson that was taught by another teacher, Ms H, who also participated in the WMCS professional development (PD) project. It was taught more than a year after the video of Mr T's lesson was taken. In the intervening time, the WMCS teachers learned about the benefits of well structured *example sets* for their students' learning in the PD. The teachers also had opportunity to work on what was a good *explanation or justification* for a particular concept or procedure, and on why the *words you use* to talk about mathematical ideas in class are important.

The context for learning more about these three key elements of teaching we had identified in research was a "Lesson Study", described briefly in Chapter 2. In a Lesson Study cycle in the project, a group of teachers worked together with a project member to plan, teach, and reflect on a lesson. The lesson was then replanned, retaught and reflected on a second time. The form of the PD is not our focus in this chapter, and we do not go into further detail. However, what is important for our purposes here is that the planning, teaching and reflecting in lesson study work were all informed by the research insights that were discussed in detail in Chapter 5, insights related to three key elements of teaching highlighted above.

Ms H's lesson was the second lesson in one of our Lesson Study cycles. We use it in this chapter to show what is possible when the planning and implementing of a lesson have been informed by insights from research. Specifically, we use Ms H's lesson plan and parts of its implementation to show what opportunities to learn are opened up when there is deliberate attention to selecting and sequencing examples; and to the language used to name and justify the mathematics in the lesson. While reading the description of the lesson we suggest you pay particular attention to the following three aspects: the opportunities for dialogue and interaction between learners and the teacher, how these were provoked and supported by the choice of the examples in the lesson, and the questions Ms H asked to build explanations for key ideas in the lesson.

THE LESSON

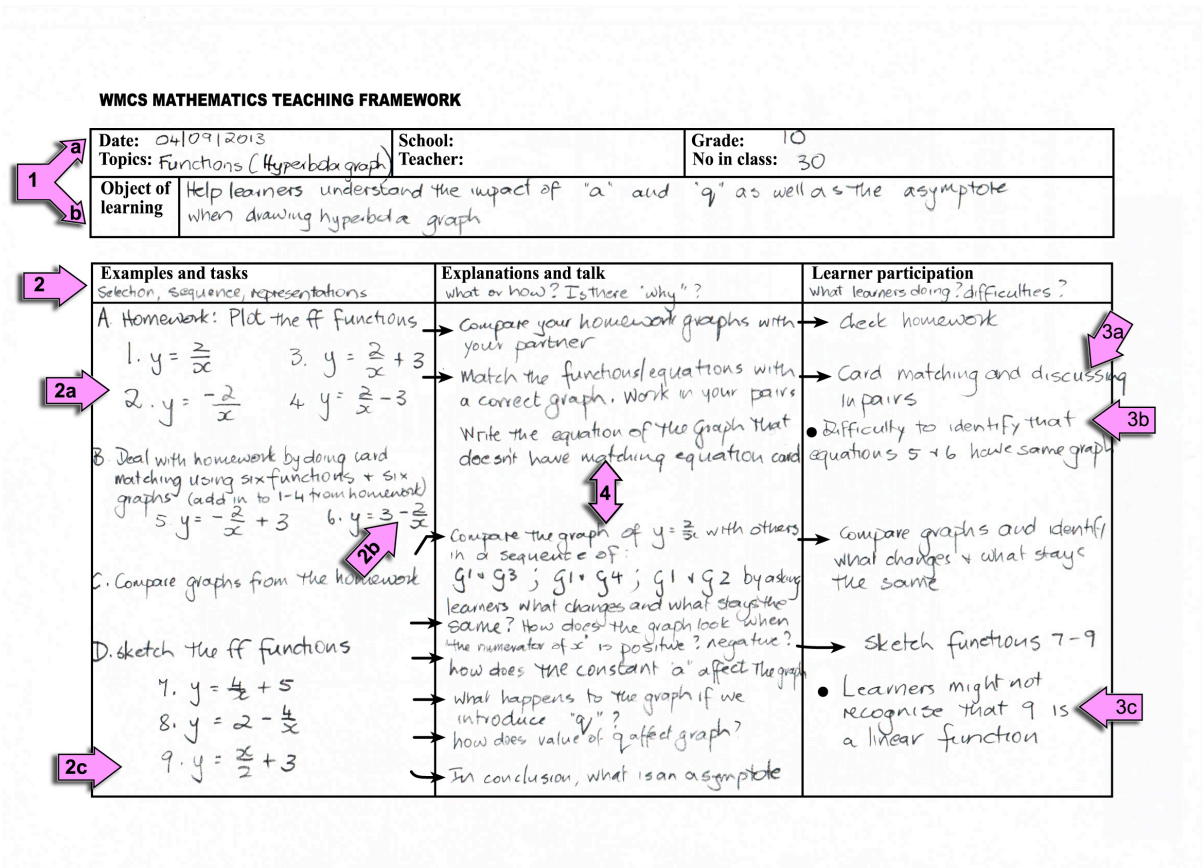
This was a Grade 10 lesson, with the goal to *"... help learners understand the influence of a and q and the asymptote on drawing the hyperbola graph".*

The Grade 10 curriculum in South Africa is focused on the parameters a and q in the general function equation $f(x) = \frac{a}{x} + q$ $(x \neq 0)$, with the horizontal shift left until Grade 11. Ms H and her colleagues in her lesson study group chose the focus: they were concerned with their learners' poor performance on questions about functions in their recent mid-year tests, and particularly the hyperbola. Our advice is that you keep the goal in mind as you continue reading about the lesson, its plan and implementation below. What support will learners need if they are to learn about the effect of the sign of a on the orientation of hyperbola graph, and of q on the horizontal asymptote of the function?

Figure 1 below shows the plan Ms H wrote for her lesson. If you look down the column on the left, you will see there were four parts to the lesson. Part A was to check the homework learners had done. The task was to plot the graphs of equations 1 – 4 in the plan on graph paper. Part B was a card matching activity with six cards each with an equation (numbers 1 – 6 on the plan below), and six cards each with a graph. The graphs, numbered G1 – G6 for our purposes in this chapter, are shown in Figure 2 below. Part C was to compare different pairs of Graphs G1 – G4, focusing on similarities and differences between the graphs in each pair. The first pair to be compared was G1 and G3. In Part D there were three more equations numbered 7 – 9. Learners were to sketch the graph of each. In this column Ms H wrote the examples she planned to use in each part of the lesson.

The next two columns in Figure 1 show what Ms H planned for learners to do with the equations and graphs in each of the four parts of the lesson, and the questions she was going to ask about the graphs and about the effects of a and q on the graph.

We focus our discussion in the rest of this chapter on selected segments of the lesson, each of which is numbered on the plan. We begin with the overall plan itself.


WMCS MATHEMATICS TEACHING FRAMEWORK

Date: 04/09/2013 Topics: Functions (Hyperbola graph)	School: Teacher:	Grade: 10 No in class: 30
Object of learning	Help learners understand the impact of "a" and "q" as well as the asymptote when drawing hyperbola graph	

Examples and tasks Selection, sequence, representations	Explanations and talk what or how? Is there 'why'?	Learner participation what learners doing? difficulties?
A. Homework: Plot the ff functions 1. $y = \frac{2}{x}$ 3. $y = \frac{2}{x} + 3$ 2. $y = -\frac{2}{x}$ 4. $y = \frac{2}{x} - 3$	Compare your homework graphs with your partner Match the functions/equations with a correct graph. Work in your pairs Write the equation of the graph that doesn't have matching equation card	check homework Card matching and discussing in pairs • Difficulty to identify that equations 5 & 6 have same graph
B. Deal with homework by doing card matching using six functions + six graphs (add in to 1-4 from homework) 5. $y = -\frac{2}{x} + 3$ 6. $y = 3 - \frac{2}{x}$		
C. Compare graphs from the homework	Compare the graph of $y = \frac{2}{x}$ with others in a sequence of: g1 & g3; g1 & g4; g1 & g2 by asking learners what changes and what stays the same? How does the graph look when the numerator of x is positive? negative?	Compare graphs and identify what changes & what stays the same
D. sketch the ff functions 7. $y = \frac{4}{x} + 5$ 8. $y = 2 - \frac{4}{x}$ 9. $y = \frac{x}{2} + 3$	how does the constant 'a' affect the graph what happens to the graph if we introduce "q"? how does value of q affect graph? In conclusion, what is an asymptote	Sketch functions 7-9 • Learners might not recognise that 9 is a linear function

Fig 1: Ms H's lesson plan

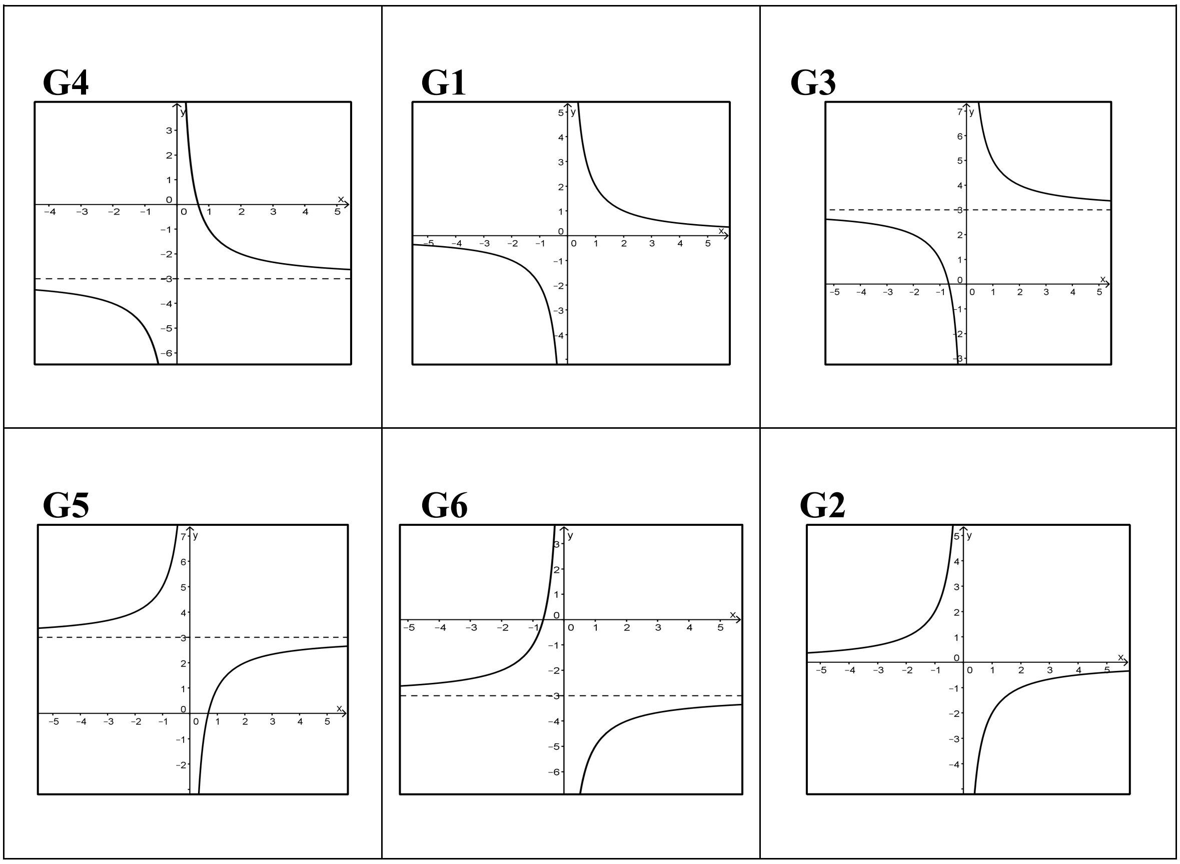


Fig 2: The graph cards for the card matching task

NOTES 1: THE FRAMEWORK AND THE OBJECT OF LEARNING

The first thing you probably noticed is the form of the lesson plan. This framework (note 1a) is what we use in the WMCS project for planning and then reflecting on lessons in our lesson study work. The framework is not a typical template for lesson planning. We know that teachers write lesson plans in many different ways. In our lesson study work, we ask teachers to present their plan within the framework template. What this requires, first and foremost, is that the lesson has an explicit goal – what we call the object of learning. What are learners to know and be able to do as a result of participating in this lesson? The planned examples and their order, as well as what kinds of explanations will be built, and what learners will do are then all developed to meet the overall lesson goal. The teacher who will teach the lesson writes these further plans into the template, as did Ms H in Figure 1. We can see her deliberate attention to examples, explanations and learner participation in the lesson.

The object of learning stated in the plan (note 1b) above was to "help learners understand the impact of *a* and *q* [on the graph] as well as the asymptote". If we look at the plan overall, we can see learners were expected to be able to sketch a hyperbola (the graph of rational function of the form $f(x) = \frac{a}{x} + q$), given its equation (i.e. given particular values for a and q), and so recognise how sign of a affects the orientation of the graph, and q its horizontal asymptote. Even prior to this, learners needed to recognise the general form of the hyperbola equation as $y = \frac{a}{x} + q$.

Let's look first at the set of examples selected for the lesson with this question in mind: how do these combine to support the lesson goal?

NOTES 2: SELECTING AND SEQUENCING EXAMPLES

Examples are key in any mathematics lesson. Indeed, when we think and talk about mathematics, it is almost impossible to do this without providing some particular examples of what we are thinking about or explaining. We can see in the lesson plan that there were nine function equations. Two of these equations were equivalent in that they represented the same function. The ninth example was of a linear function. Each example appeared in both equation and graphical form in the lesson. Was this selection of examples and their sequencing in the lesson supportive of learning about the impact of a and q in the equation on the rational function? In other words, did these examples provide opportunities for learners to notice the relationship between the equation and the hyperbola, the graph of the function $f(x) = \frac{a}{x} + q$?

We think they did! So let's look at how we come to this conclusion. We start with examples 1 – 4 (note 2a), where the value of a is either 2 or -2. Keeping 2 constant, and changing the sign of 2 brings into focus the relationship between the sign, and the orientation of the graph. In this same set of four examples, the value of q is either 0 or ± 3. Here too it becomes possible to focus on the vertical shift of the graph. This attention to *varying some aspects of the general equation, and keeping others invariant* is a powerful way of seeing *similarity* across the four equations and their related graphs. In particular, it helps in distilling the *general* form of the equation of the hyperbola, as the graphs of these functions are called, and in discerning the impact of parameters on the shape and position of the curve.

Equally powerful as devices for learning were equations 5 and 6, and graphs G5 and G6 for the card sorting activity (Note 2b). There was no equation to match graph G6, and equation 6 matched graph G5. In order to decide which graph matched equation 5, and then what was the equation for Graph 6, learners had to focus on the values of a and q and try to figure out how these related to particular graphs. Firstly, equation 6 leads to recognition that the order of the terms does not matter for the shape and location of the graph. Secondly, learners needed to construct the equation for graph G6 to confirm that its matching equation was not in the pack. What is interesting is that these additional examples were deliberately selected. We can see in the "learner participation" column that Ms H anticipated that some learners would have difficulties (Note 3b), and that they might disagree with each other as they tried to match these particular cards.

The final three examples 7-9 (Note 2c) provided opportunity to see whether the learners were able to sketch a graph from a given equation. One of these equations reversed the order of terms, thus reinforcing example 6, and another one was linear (example 9). In this latter example, Ms H moved beyond the question of what was the same and different about particular pairs of hyperbola graphs and included a “non-hyperbola” given by a linear equation with a fraction. This *contrasting e*xample brings attention to things that might appear “the same” but are different. Ms H anticipated that some learners would have difficulties with example 9, precisely because she was aware that some of them would focus on appearance. This would create another opportunity for her to use disagreement amongst learners to focus their attention on those aspects of mathematical forms that are relevant in the given context.

Across the examples, there was also possibility for focusing on parameters a and q separately and then together. In algebraic expressions, a number of elements may vary at the same time. *Varying the parameters separately and then together is helpful for learning.* In our PD sessions, teachers have discussed the question of whether varying parameters simultaneously causes confusion. They explained why they tended to deal with only one parameter at a time in their teaching.

The nine examples in the lesson had *q varying while a remained invariant, and vice versa.* Together, they offered opportunities for learning about the impact of a and q on the hyperbola graph. The *contrasting* example 9 focused the learner's attention on the placement of the variable x in the equation of the rational function.

Constructing sets of examples is indispensable in any lesson. Attending to what is *invariant and similar helps in generalisation.* Seeing *counter examples, or non-examples* also offers opportunity for seeing what is not included in the class of examples being discussed. In short, it is of utmost importance to *build generalisations deliberately, by considering each aspect separately, creating contrasts and eventually also altering more than one element at a time.* How these examples are sequenced is also of great significance. All these are important moves towards realising the lesson goal. Ms H's lesson provides an interesting illustration of how this can be done.

Of course, there is always a difference between a plan and its implementation in the classroom. So let's move on to what actually happened when learners were presented with this range of examples.

NOTES 3: LEARNERS' PARTICIPATION IN THE LESSON

Let's start by looking at how Ms H *planned* for learner participation in the lesson. The note in Part B in right hand column of Figure 2 (Note 3a) shows that the learners were to work in pairs, whereas the task was to match hyperbola graphs and equations. In Part C they would offer suggestions about the properties of the hyperbola by focusing on what is the same and what is different across Graphs 1 – 4 and their matching equations. In part D they would practice sketching graphs for given equations.

In Ms H's notes in each of these three parts we can see her intention to get learners to also talk about what they were doing. In part B, pairs of learners would share a pack of cards. They would need to *discuss and come to an agreement on which cards matched one another*. Ms H expected learners to *disagree w*ith each other about which graph matched equation 6. She realized, therefore, that at this point there would be a need for *dialogue* (Note 3b). In part C learners would need to suggest, and so *describe in words*, what was similar and what remained different across the graphs. In part D she anticipated that learners would make a mistake with equation 9, and that they would treat it as a hyperbola. So, here too there would be *opportunity for disagreement and discussion* (Note 3c).

This leads to a note on the third aspect of teaching mentioned in the introduction to this chapter – the language used by Ms H and by her learners to name and justify mathematics.

NOTES 4: NAMING AND JUSTIFYING MATHEMATICS IN THE LESSON

Throughout the lesson, Ms H listened carefully to what learners said. She did this as she moved between pairs working together on the card matching activity. She also listened carefully to learners' contributions during whole class discussion. Listening to learners is critical in any mathematics classroom, and particularly so in a multilingual classroom. You really need to hear what learners are saying and how they express themselves mathematically and in English.

Ms H revoiced many of learners' utterances. What we mean by 'revoicing' is that Ms H frequently repeated what learners had said, modifying their expressions but still trying to preserve the gist of their ideas. While doing so, she paid special attention to words used in describing the mathematical ideas in focus. The two examples below are taken from part C of the lesson where learners were to describe what was similar and what was different about graphs G1 and G3 and then G1 and G4. In each example we provide first what a learner said, and then Ms H's repeating and then revoicing of the learner's expression, with the latter italicised.

Example 1

L: Ma'am in number one, you can see this Ma'am, so when you compare it to number three it moved three spaces up

Ms H: Okay so she says what changed is that the graph moved, it moved up by three spaces. We can count (point to the graph on the board). *The graph shifted three units up on the Y-axis.*

Example 2

L: Another change is the symmetry. The one symmetry is y is equals to x, the one with the zero asymptote. The symmetry will be y equals to x and then for the second one it will be y is equals to x plus 3 because the asymptote is three.

Ms H. Okay she noted another thing that changed was the symmetry, because this one (pointing to G1), *here the line of symmetry is given by the equation y is equal to x. But here (pointing to G3), the graph has shifted three units up on the Y-axis. So the line of symmetry now is the equation y is equal to x plus 3. The y-intercept of this line is now y equals 3, whereas here (pointing to G1) it is y equals 0. So the line of symmetry has changed.*

In the discussion contrasting the different graphs, Ms H brought out for public discussion various properties of the hyperbola such as the domain, range, symmetry, and the orientation of the graph on the plane relative to the value of a. Managing such whole class discussion involves ongoing on the spot decision-making about informal and formal mathematical talk. One also has to decide whether to present full-fledged mathematical explanations or look for reasonable shortcuts. Of course there are inevitable tensions in a lesson where time constraints require choices about what to say, when, how and why. So let us look in more detail now at how Ms H introduced and encouraged formal language use and how, in her whole class discussions, she provided mathematically informed rationales for mathematical ways of doing things. We focus on the two episodes in which she worked with learners on examples 6 and 9, the ones she expected to provoke disagreement. From these episodes one can learn how to work with what learners say, and particularly when this talk is vague or relies on appearances rather than mathematical relations.

- **Describing the structure of the general equation and justifying this when different forms are presented**

Just as she had anticipated, learners had difficulties with equation 6 and graph G6. Perhaps they were reluctant to match two seemingly different equations $y = 3 - \frac{2}{x}$ and $y = \frac{-2}{x} + 3$ with one graph. They might also be unprepared for the situation of a graph card with no matching equation card. Ms H noticed all this as she moved around the class and watched what the pairs of learners were doing and saying. She brought the class together to look at the problematic cards. Through probing questions and then revoicing learner offerings in whole class discussion, she reached agreement with learners that "*equation 6 had a horizontal asymptote, the equation of which is y = 3*" (see 2a above) and then asked:

Ms H: So let's look at all the other graphs where the horizontal asymptote has the equation y = 3 ...

All the graphs were sketched on the chalkboard, and learners quickly called out their responses, some suggesting "number three" and others " number five". She asked the whole class:

Ms H: Do you all agree that these graphs (pointing to G3 and G5 on the board) *both have the same horizontal asymptote*?

and then continued:

Ms H: ... So that means you are saying that this equation (writing the equation $y = 3 - \frac{2}{x}$ next to Graph G5) is the same as this equation (writing $y = \frac{-2}{x} + 3$ also alongside G5). This means they represent the same function, and so will have the same graph and the same horizontal asymptote.

Learners were shouting out, some saying yes, others saying no.

Ms H. Ok now let's look at these equations. We have y equals to negative two over x plus three and the other one is y equals to three minus two over x ... (pointing to the two equations she has written on the board)

One learner puts up her hand, and on invitation to speak says:

L1: Mam ... here the first one (referring to the equation $y = \frac{-2}{x} + 3$) negative two over x it's the, it's the steep ...and the uh... plus three is the asymptote. This side (referring to the equation $y = 3 - \frac{2}{x}$) three is the steep and negative two over x is the asymptote. So they change everything!

To which other learners called out both disagreement and some agreement.

Ls: Yes... no... no... asymptotes

Ms H: Ok ... now let's look at these two equations. ...

She rewrote the equations $y = 3 - \frac{2}{x}$ and $y = \frac{-2}{x} + 3$ one below the other and pointed clearly at each term as she continued:

Ms H: ... what do we have here (pointing to the first equation)? Instead of starting with negative two over x and then adding 3, we have started with three, and then we add negative two over x. So those two are exactly the... (she pauses)

Ls: Same ... (called out in chorus)

First we need to point out that this isn't a full extract – we have not included all learners' inputs, particularly the collective calling out. We want to highlight how attention to example 6 has brought some learners' confusion into focus for the whole class. We also wish to highlight the words Ms. H used to express this. It was only when she listened to L1 that she could appreciate the difficulty some learners were having, and she could thus understand why there had been disagreement. The association of the 'asymptote' and the 'steep' with the *order* of the term in the equation and not the term itself was a surprise. So how did Ms H work with learners to justify that the two equations in focus represented the same function?

We see that she talked about the graph with a *horizontal asymptote with equation y equals 3*, and then shifted to talk about the equations. In this way, she focused on an algebraic structure and form with which learners were familiar. She then rewrote the equations to show they were simply the same, taking as shared with others in the class, that on the basis of commutative property, she transformed the first equation into the second, thus showing their full equivalence. How learners talked about what they were looking at takes us back to some of the naming and justifying in Mr T's lesson, where focus was on appearance, or visual form and not on properties of mathematical objects (recall his comparison of inequality sign to a crocodile's mouth or the vertex of the parabola graph as a cupped hand). Ms H shifted learners' attention through the words she used. She named what they were looking at differently, and turned learners' attention away from appearance or position of the terms, and onto the whole expression, transforming it to show the equivalence.

This episode brings to light the opportunity Ms H created for learner ideas (and errors) to enter the whole class talk. This was also an opportunity for herself to take these ideas up and to provide some justification for why the two equations were "the same". Supporting learners whose language indicates misunderstanding requires a teacher to carefully attend to how words are used and how and when mathematical ideas are justified. In Chapter 5 we referred to this to as the *explanatory talk* in a lesson.

What else can we learn from the dialogue around the equation of the straight line in example 9 and from Ms H's naming and justifying as she interacted with learners on this?

Naming and justifying the graph in relation to its equation form

While moving between learners who were sketching graphs of the equations in examples 7 – 9, Ms H noticed that for equation 9, most of them had drawn a hyperbolas. Indeed, one learner whom she invited to the board to sketch and explain their graph to the class, sketched a hyperbola. Ms H then asked the class whether they agreed with the sketch and whether anyone had a "different graph". Many answered "no", but a few called out "yes". To provoke a discussion, Ms H identified one of those learners who disagreed and said:

Ms H: Ok, can you please draw it for us … Yes go and draw it quickly

The learner sketched the line $y = x$, and other students called out:

Ls: No but it's wrong ... right ...
Ms H: Ok now she's, she's drawn a straight line... She's saying it's a straight line, do you agree?
Ls: Yes
Ls: No
L3: Yes ma'am it's a straight line...
Ms H: So it is ... do you agree that it's a straight line? Is this a hyperbola or it's a linear function? (pointing to the equation)
Ls: A linear function ... a linear...
Ms H: This one's a linear function. Now look at the equation … is it a hyperbola or is it linear?
Ls: Linear...
Ms H: Why do you say it's linear?
Ls: Ma'am because the x is in... the x is in the numerator not the denominator
Ms H: Yes, x is in the numerator not the denominator … in the equation of the hyperbola, x is in the denominator. These terms are different. x over two is a half times x, and two over x is two times one over x (and she wrote: $\frac{x}{2}$ is $\frac{1}{2}\times x$, and $\frac{2}{x}$ is $2 \times \frac{1}{x}$)

At this point Ms H asked the class to compare the graph drawn with the equation, and whether these are the same function. A learner then came to the board and drew the correct linear graph, and explained that the line had a y-intercept of 3.

And so here too we see that properly chosen examples may bring into full relief some common errors, evoke discussion and create an opportunity to explain the correct solution. Ms H revoiced the learners' description of the difference between the two equations focusing as they did on visual forms, and so with attention to location of x in the fraction. She then went on to explain why the two fractions were different by rewriting $\frac{x}{2}$ as $\frac{1}{2} \times x$, and $\frac{2}{x}$ as $2 \times \frac{1}{x}$.

Ms H's explanation here is instructive. She first revoiced the justification based on appearance, before she explained the difference between the two terms. Ms H brought this particular episode up for discussion after the lesson as she was not convinced that her second explanation was helpful for learners, and asked the other teachers in the Lesson Study group who had observed the lesson how else she could have spoken about this. Some teachers thought her explanation was fine. Others suggested that the problem is learners still don't appreciate that these terms stand for numbers, and that it might have helped to emphasise that since $\frac{x}{2}$ and $\frac{2}{x}$ were numbers, unless x was equal to 2, these were different numbers. We can see here that in the Lesson Study reflection time the participating teachers were able to discuss further how naming and justification worked in the lesson.

Conclusion

We return now to our introductory section, and to the question of why we chose this lesson. We wanted to show the great importance of deliberate attention to selection and sequencing of examples and to the language used in talking about them. Ms H's lesson sheds light on the opportunities for learning she opened up in this way while discussing rational functions of the form $y = \frac{a}{x} + q$, and their relationship to the graph. In the WMCS PD, teachers have found it very useful to reflect on their choice of examples, and on the question of how to make such choices so as to bring the object of learning into focus. Inevitably in teaching there are choices about when to bring some things into focus and background others. Thinking about and planning for an example set in a lesson, and how this supports what you want learners to learn is productive. A further reason why we chose Ms H's lesson is that it brings to the fore the complexities of stimulating dialogue in a lesson. This requires listening to what learners say, and then building on this to focus learners' attention on how words are used and how on how we justify ourselves mathematically. We hope this has been a lesson to learn from, particularly about example sets, explanatory talk and learner participation, the key elements of teaching in the WMCS teaching framework.